Published by Institute of Physics Publishing for SISSA
Received: *February 21, 2006*
Accepted: *March 17, 2006*
Published: *April 6, 2006*

Towards a string bit formulation of $\mathcal{N} = 4$ super Yang-Mills

Luis F. Alday,[a] **Justin R. David,**[bd] **Edi Gava**[bc] **and Kumar S. Narain**[b]

[a] *Institute for Theoretical Physics and Spinoza Institute, Utrecht University 3508 TD Utrecht, The Netherlands*
[b] *High Energy Section*
The Abdus Salam International Centre for Theoretical Physics
Strada Costiera, 11-34014 Trieste, Italy
[c] *Instituto Nazionale di Fisica Nucleare, sez. di Trieste and SISSA*
Italy
[d] *Harish-Chandra Research Institute*
Chhatnag Road., Jhunsi, Allahabad 211019, India
E-mail: L.F.Alday@phys.uu.nl, justin@hri.res.in, gava@ictp.trieste.it, narain@ictp.trieste.it

Abstract: We show that planar $\mathcal{N} = 4$ Yang-Mills theory at zero 't Hooft coupling can be efficiently described in terms of 8 bosonic and 8 fermionic oscillators. We show that these oscillators can serve as world-sheet variables, the string bits, of a discretized string. There is a one to one correspondence between the on shell gauge invariant words of the free Y-M theory and the states in the oscillators' Hilbert space, obeying a local gauge and cyclicity constraints. The planar two-point functions and the three-point functions of all gauge invariant words are obtained by the simple delta-function overlap of the corresponding discrete string world sheet. At first order in the 't Hooft coupling, i.e. at one-loop in the Y-M theory, the logarithmic corrections of the planar two-point and the three-point functions can be incorporated by nearest neighbour interactions among the discretized string bits. In the $SU(2)$ sub-sector we show that the one-loop corrections to the structure constants can be uniquely determined by the symmetries of the bit picture. For the $SU(2)$ sub-sector we construct a gauged, linear, discrete world-sheet model for the oscillators, with only nearest neighbour couplings, which reproduces the anomalous dimension Hamiltonian up to two loops. This model also obeys BMN scaling to all loops.

Keywords: AdS-CFT Correspondence, Supersymmetric gauge theory, Lattice Integrable Models.

JHEP04(2006)014

Contents

1. Introduction **1**

2. $\mathcal{N}=4$ Y-M spectrum in terms of oscillator variables **5**
- 2.1 The conformal algebra 5
- 2.2 The oscillator construction 7
- 2.3 Partition function 11

3. String overlap at $\lambda = 0$ **14**
- 3.1 Single bit overlap 14
- 3.2 Two-point functions 19
- 3.3 Three-point functions 22

4. String overlap at one-loop in λ **25**
- 4.1 Logarithmic corrections at one loop 28
- 4.2 Structure constants in the $SU(2)$ subsector 33

5. Anomalous dimensions for the $SU(2)$ sector. **37**
- 5.1 Anomalous dimensions to two loops 38

6. Conclusions **42**

A. Notations and conventions **43**

B. Properties of the transform U **45**

C. The $PSU(2,2|4)$ algebra **45**

D. Bit string overlap and three-point functions **46**

E. Anomalous dimensions at three loops **48**

F. Properties of bilinears in the $SU(2)$ sector **51**

1. Introduction

Among all the examples of field theories dual to a string theories, the most precise and well studied is the case of the duality between $\mathcal{N} = 4$ Yang-Mills theory in four dimensions with gauge group $SU(N)$ and type IIB string theory on $AdS_5 \times S^5$ [1–4]. In units of the radius of AdS, the string length is related to the 't Hooft coupling of the gauge theory by

$$\alpha' = \frac{1}{\sqrt{\lambda}} = \frac{1}{\sqrt{g_{\rm YM}^2 N}}, \qquad G_N = \frac{1}{N^2}, \tag{1.1}$$

where $g_{\rm YM}$ is the Yang-Mills coupling constant, α' the string length measured in units of the radius of AdS. G_N is the Newton's constant which is the effective string loop parameter.

This duality has been tested in the supergravity regime on the string theory side, which, according to (1.1) corresponds to strong 't Hooft coupling and large N limit on the Y-M side. For applications of the duality to physically more interesting cases, one would like to be able to control the weak coupling regime in the gauge theory, which requires understanding string theory on $AdS_5 \times S^5$ at strong sigma-model coupling. The problem is that, at present at least, type IIB string on $AdS_5 \times S^5$ has not been fully quantized. This puts strong limitations to the duality tests and to the applications of the duality to physically relevant situations, confining them to the regime when type IIB string theory can be approximated by the corresponding supergravity or to sectors which involve large quantum numbers [5], for which semiclassical quantization is amenable. Early tests of the duality relied on protected quantities, like the conformal dimensions of chiral primaries of the Yang-Mills theory and the spectrum of Kaluza-Klein supergravity states on $AdS_5 \times S^5$ [2]. Their three-point functions were also shown to agree with the three-point couplings evaluated from the supergravity action [6 – 8]. At present there are no known means to extract information regarding the spectrum of string states or their correlation functions from the sigma model on $AdS_5 \times S^5$, other than in the plane wave limit [9][1] . The field theory, on the other hand, is best understood at weak coupling, in particular, as a perturbative expansion in the 't Hooft coupling. This has led to many efforts in trying to rewrite the spectrum of free $\mathcal{N} = 4$ Yang-Mills theory as a spectrum in a string theory [13 – 15]. There has also been an effort at reconstructing the string theory world-sheet by rewriting the correlation function of gauge invariant operators of the free theory as string-like amplitudes in AdS [16 – 18].

In this paper we formulate $\mathcal{N} = 4$ Yang-Mills theory at weak coupling in terms of a discrete world sheet made of bits, starting from the free point and then switching on the gauge coupling perturbatively. There are two main motivations for a discrete world-sheet description of the Yang-Mills at weak coupling. We detail these below.[2]

Motivations for a string bit description. Let us again recall the map between the basic parameters of the string theory and $\mathcal{N} = 4$ Yang-Mills. From (1.1) we see that in the free limit, the string length measured in units of the radius of AdS becomes infinite and the string essentially becomes tensionless, there is no coupling between neighbouring points on the string. Therefore the string breaks up into non interacting bits held together only by the $L_0 = \bar{L}_0$ constraint [19]. This can be seen more precisely using the Green-Schwarz type IIB superstring moving in $AdS_5 \times S^5$ background. In the light cone gauge

[1] See [10 – 12] for developments in the use of spinning strings to to study the quantization of the sigma model on $AdS_5 \times S^5$.

[2] Throughout this paper we will work in the planar limit of the Y-M theory.

the Hamiltonian which generates evolution in the light-cone time variable x^+ is given by [20 – 22]

$$\mathcal{H} = \int d\sigma \frac{1}{p_+}\left(P^2 + \frac{T^2}{r^4}(\partial_\sigma X)^2\right). \tag{1.2}$$

Here we have written down only the bosonic co-ordinates, $T \sim 1/\alpha' = \sqrt{\lambda}$ the tension of the string, P stands for the momenta of the bosonic co-ordinates and X the position. r refers to the radial distance in AdS, and we have used the Poincaré co-ordinates of AdS to write the sigma-model action. The point to be emphasized is that the terms with derivatives in the σ co-ordinates of the world sheet are suppressed by powers of the string tension. In the tensionless limit we can neglect these terms and the string breaks up into bits of non-interacting particles held together only through the level matching constraint $L_0 = \bar{L}_0$.

The second motivation to think of the string theory dual to free Yang-Mills in terms of bits arises from examining the AdS/CFT duality in the plane wave limit [9]. BMN set up a dictionary between single trace Yang-Mills operators of large R-charge and states in the plane wave string theory. An example of such a correspondence is the following

$$a_n^{\dagger i} a_{-n}^{\dagger j}|0, p_+\rangle_{\rm l.c} \leftrightarrow \frac{1}{\sqrt{JN^{J+2}}}\sum_{l=1}^{J} \mathrm{Tr}\left(\phi^i z^l \phi^j z^{J-l}\right) e^{\frac{2\pi i n l}{J}}. \tag{1.3}$$

Here i, j are directions which correspond to the $SO(4) \subset SO(6)$ part of the R-symmetry. z is the complex scalar which belongs to the Cartan and has R-charge unity. $a_n^{\dagger i}$ refers to the oscillator modes of the plane wave string theory. Though the above dictionary was set up for operators with large R-charge, $J \sim \sqrt{N}$, let us imagine extrapolating the dictionary for finite J but with $g_{\rm YM} = 0$. As $g_{\rm YM}$ is set to zero, one expects that there is no renormalization of the states. For finite J from (1.3) it is clear that the Yang-Mills state is invariant for $n \to n + J$, thus we arrive at the conclusion that the modes of the string theory are truncated at order J. This implies that the world sheet is discrete, in fact the number of bits (or points) of the world sheet is of the order of the length of the Yang-Mills operator. Furthermore, it is easy to see from (1.3) that the cyclicity of trace translates to the level matching condition on the string side. In fact this discrete nature of the world sheet has already been noticed by [9, 23 – 27]

From the above discussion, it would seem that one attempt to capture some features of the string dual for free $\mathcal{N} = 4$ Yang-Mills is just to discretize the plane wave string theory. Such an attempt runs into difficulties, as the plane wave symmetry algebra is a contraction of the symmetry algebra of $AdS_5 \times S^5$. We illustrate this by considering the dual of operators which are in the traceless symmetric representation of $SO(6)$, the chiral primaries. If the highest weight state of this representation has R-charge J, then one can suppose the string dual is the vacuum made of J bits. According to the BMN dictionary the other components then should be obtained by repeated action of the creation of zero modes $a_0^{i\dagger}$. a_0^i's just lower the J charge of the vacuum state. From the correspondence with the Yang-Mills state we know that number of $a_0^{i\dagger}$ cannot exceed J. This constraint has to be imposed on the discrete plane wave theory. Such constraints make it difficult to formulate the discrete theory in terms of plane wave oscillators.

In this paper we will introduce oscillators which naturally capture the symmetries of $AdS_5 \times S^5$ geometry. We show that planar $\mathcal{N} = 4$ Yang-Mills theory at zero 't Hooft coupling can be efficiently described in terms of 8 bosonic and 8 fermionic oscillators. These oscillators can serve as the discrete world-sheet variables of a string bit formulation of the Yang-Mills theory. We show that there exists a one to one correspondence between the on-shell, gauge invariant words of the free Yang-Mills with the Hilbert space of these oscillators, together with a local $U(1)$ gauge symmetry and the cyclicity constraint. An universal feature of any string theory is that the interactions are described by the delta-function overlap of strings. In fact the structure constants of gauge invariant words, which in the planar limit are proportional to $1/N$, should emerge from the joining or splitting of the strings. We will show that the planar, two-point and three-point functions are obtained by the simple delta-function overlap of the corresponding discrete world-sheet.

We then turn on the 't Hooft coupling. From (1.1) we see that turning on λ renders the string tension finite, thereby introducing interactions between the bits. At one-loop in λ and in the planar limit only nearest neighbour bits would interact. We will show that nearest neighbour corrections to the global $SO(2,4)$ charges in the string bit formulation reproduces the logarithmic divergences of the one-loop corrected two-point functions and the three-point functions of the Yang-Mills theory. For the $SU(2)$ sub-sector we shown that the symmetries of the bit picture are sufficient to determine the one-loop corrected structure constants evaluated in [28 − 30], upto an overall coefficient.

We then focus on the anomalous dimension Hamiltonian for the $SU(2)$ sub-sector of the theory. Here the local $U(1)$ gauge symmetry of the bit picture is used to construct a gauged linear model of the corresponding oscillators. This model has only nearest neighbour couplings, but it reproduces the anomalous dimensions in this sub-sector to two loops and obeys BMN scaling to all loops.

This paper is organized as follows. In section 2 we introduce the oscillator variables and show the spectrum of the free Yang-Mills theory is identical to the Hilbert space of these oscillators once the $U(1)$ gauge constraint and the cyclicity constraint are taken into account. In section 3 we construct the 2-string and 3-string overlap at $\lambda = 0$ and show that they reproduce the two-point functions and the three-point functions of all gauge invariant words of the Yang-Mills theory. In section 4 we show that logarithmic corrections in two-point functions and three-point functions can be reproduced due to nearest neighbour corrections to the global $SO(2,4)$ generators in the bit picture. We then show that the structure constants in the $SU(2)$ sub-sector are entirely determined, upto an overall constant, from the symmetries of the bit picture. In section 5 we introduce a gauged, linear, discrete world-sheet model for the oscillators corresponding to the $SU(2)$ sub-sector, having only nearest neighbour interactions. This model reproduces the anomalous dimensions up to two loops. It also obeys BMN scaling to all loops. Section 6 contains our conclusions, appendix A contains the notations and conventions adopted in this paper. Appendices B and C contain details regarding the oscillator variables. Appendix D works out a few examples of three-point functions at $\lambda = 0$ and compares them to the three string overlap. Appendix E introduces a gauged oscillator model with non-nearest neighbour interaction which reproduces the anomalous dimensions up to three loops. Appendix F contains details of the oscillator algebra necessary for evaluating the anomalous dimension Hamiltonian from the gauged oscillator model.

2. $\mathcal{N}=4$ Y-M spectrum in terms of oscillator variables

The main objective of this section is to write the spectrum of free $\mathcal{N}=4$ Yang-Mills entirely in terms of eight bosonic and eight fermionic oscillators. These oscillators will serve as the discretized world sheet variables of a string theory which we construct to describe the Yang-Mills theory. We will refer to these oscillators as bits.

The symmetries of $AdS_5\times S^5$ form the supergroup $PSU(2,2|4)$, the bosonic component of this supergroup is given by $SO(2,4)\times SO(6)$. The $SO(2,4)$ corresponds to the conformal group in the $\mathcal{N}=4$ super-Yang Mills in four dimensions, while $SO(6)$ to its R-symmetry. We will first discuss the conformal algebra and the strategy we will adopt in classifying the representations of the conformal group. Then we use the oscillator variables introduced in [31, 32] [3] to write down all the generators of $PSU(2,2|4)$. We then construct all the letters and the single trace gauge invariant words of $\mathcal{N}=4$ Yang-Mills using the oscillator variables. We evaluate the partition function of the single letters and the single trace gauge invariant words using the oscillator variables and show that they are in one to one correspondence with the gauge theory operators at zero coupling modulo equations of motion and Bianchi identities.

2.1 The conformal algebra

The conformal group in four dimensions $SO(2,4)$ is generated by the Lorentz generators $M_{\mu\nu}$, the four momentum P_μ, the generators of special conformal transformations K_μ, $(\mu,\nu=0,1,2,3)$ and the Dilatation generator D. The algebra of the generators is given by

$$\begin{aligned}
[M_{\mu\nu},M_{\rho\sigma}] &= i(\eta_{\nu\rho}M_{\mu\sigma}-\eta_{\mu\rho}M_{\nu\sigma}-\eta_{\nu\sigma}M_{\mu\rho}+\eta_{\mu\sigma}M_{\nu\rho}),\\
[P_\mu,M_{\rho\sigma}] &= i(\eta_{\mu\rho}P_\sigma-\eta_{\mu\sigma}P_\rho),\\
[K_\mu,M_{\rho\sigma}] &= i(\eta_{\mu\rho}K_\sigma-\eta_{\mu\sigma}K_\rho),\\
[D,M_{\mu\nu}] &= [P_\mu,P_\nu]=[K_\mu,K_\nu]=0,\\
[P_\mu,D] &= iP_\mu, \quad [K_\mu,D]=-iK_\mu,\\
[P_\mu,K_\nu] &= 2i(\eta_{\mu\nu}D-M_{\mu\nu}).
\end{aligned} \tag{2.1}$$

here $\eta_{\mu\nu}=$ diag $(-,+,+,+)$. The correspondence with the $SO(2,4)$ generators is made with the identifications

$$M_{\mu5}=\frac{1}{2}(P_\mu-K_\mu), \quad M_{\mu6}=\frac{1}{2}(P_\mu+K_\mu), \quad M_{56}=-D, \tag{2.2}$$

then these generators satisfy the $SO(2,4)$ algebra given by

$$[M_{AB},M_{CD}]=i(\eta_{BC}M_{AD}-\eta_{AC}M_{BD}-\eta_{BD}M_{AC}+\eta_{AD}M_{BC}). \tag{2.3}$$

Here $A,B,\ldots=0,1,2,3,5,6$ and $\eta_{AB}=\mathrm{diag}(-,+,+,+,+,-)$. Note that the directions $0,6$ are time like and the directions $1,2,3,5$ are space like.

[3] The latter reference contains a concise review of the oscillators variables.

A convenient way to classify representations of $SO(2,4)$ is to use the maximal compact subgroup $SO(2) \times SO(4) = U(1)_E \times SU(2)_L \times SU(2)_R$. The compact generators correspond to

$$\begin{aligned} L_m &= \frac{1}{2}(\frac{1}{2}\varepsilon_{mnl}M_{nl} + M_{m5}), \qquad : SU(2)_L, \\ R_m &= \frac{1}{2}(\frac{1}{2}\varepsilon_{mnl}M_{nl} - M_{m5}), \qquad : SU(2)_R, \\ E &= M_{06} = \frac{1}{2}(P_0 + K_0), \qquad : U(1)_E, \end{aligned} \tag{2.4}$$

where m, n, l refer to the space like directions $1, 2, 3$. From these generators it is easy to see that the conformal algebra has a decomposition as $SO(2,4) \to L^+ \oplus L^0 \oplus L^-$ such that the following commutation relations hold

$$\begin{aligned} [L^0, L^\pm] \subset L^\pm, \qquad [L^+, L^-] \subset L^0, \\ [E, L^\pm] = \pm L^\pm, \qquad [E, L^0] = 0. \end{aligned} \tag{2.5}$$

We can now construct unitary representations from the state $|\Omega\rangle$ with quantum numbers (j_L, j_R, E) and annihilated by the elements of L^-. The representations are obtained by the action of the raising operators L^+ on the state $|\Omega\rangle$. E will have to be bounded from below for a physically relevant unitary representation. We denote the space of these representations by $\mathcal{H}_1$.

In the conformal invariant $\mathcal{N} = 4$ Yang-Mills, one usually represents the action of the conformal group on gauge invariant operators say at $x = 0$. The stability group at $x = 0$ is generated by the Lorentz generators $M_{\mu\nu}$ which form $SL(2, C)$, the dilatations D and the special conformal generators K_μ. The primary fields are those which are annihilated by K_μ, they carry the $SL(2, C)$ quantum numbers (j_M, j_N) and the conformal dimensions Δ. The generators of $SL(2, C)$ can be written down as two $SU(2)$'s given by

$$\begin{aligned} M_m &= \frac{1}{2}(\frac{1}{2}\epsilon_{mnl}M_{nl} + iM_{0m}), \\ N_m &= \frac{1}{2}(\frac{1}{2}\epsilon_{mnl}M_{nl} - iM_{0m}), \\ D &= -M_{56}. \end{aligned} \tag{2.6}$$

It is sufficient to restrict our attention to only primaries and generate all the secondaries by the action of momentum generator P_μ. We denote the space of these representations by $\mathcal{H}_2$.

We now wish to find an isomorphism between $\mathcal{H}_1$ and $\mathcal{H}_2$. Comparing (2.4) and (2.6) it is easy to see if there is a flip of the 5-axis to i times the 0-axis the generators of L_m and R_m of (2.4) go over to M_m and N_m (2.6) and E goes over to $-iD$. Therefore to perform this rotation one needs to rotate by an angle $\pi/2$ in the $0-5$ plane with a factor of i. This is performed by the following operation

$$U = \exp\left(\frac{\pi}{2}M_{05}\right) = \exp\frac{\pi}{4}(P_0 - K_0). \tag{2.7}$$

Note that in the above we do not have a factor of i in the exponent, this takes care of the fact that we need to rotate the 0 axis to i times the 5 axis. A detailed proof will be given

JHEP04(2006)014

in the appendix B. The transformation U has the following properties

$$U^{-1}K_\mu U = L^-, \quad U^{-1}P_\mu U = L^+, \quad U^{-1}DU = iE. \tag{2.8}$$

Our strategy to obtain primary fields at the origin in $\mathcal{H}_2$ will be to start with the state $|\Omega\rangle$ in $\mathcal{H}_1$ which is annihilated by all L^- and then perform the transformation $U|\Omega\rangle$. It is now easy to see that using the first equation in (2.8), we have $K_\mu U|\Omega\rangle = 0$. Therefore we can identify $U|\Omega\rangle$ with a gauge invariant operator at $x = 0$. Now to translate it to an arbitrary position x, we further perform the transformation $\exp(ixP)U|\Omega\rangle$.

2.2 The oscillator construction

The conformal group. We now discuss the method of constructing unitary infinite dimensional representation of $SO(2,4)$ using eight bosonic oscillators. With this aim we organize the eight bosonic oscillators as into four complex oscillators transforming in the spinor representation of $SO(2,4)$, this given by.

$$\psi = \begin{pmatrix} a^{\dot 1} \\ a^{\dot 2} \\ -b_1^\dagger \\ -b_2^\dagger \end{pmatrix}. \tag{2.9}$$

Here the oscillators $a_{\dot\gamma}$ and b_γ obey the following commutation relations

$$\begin{aligned} &[a^{\dot\gamma}, a_{\dot\delta\dagger}] = \delta^{\dot\gamma}_{\dot\delta}, \qquad [b^\gamma, b_{\delta\dagger}] = \delta^\gamma_\delta, \\ &[a^{\dot\gamma}, b^\gamma] = 0 \qquad [a^{\dot\gamma}, b^\dagger_\gamma] = 0, \qquad \gamma, \delta, \dot\gamma, \dot\delta = 1, 2. \end{aligned} \tag{2.10}$$

[4] The action of $SO(2,4)$ on the Fock space of these oscillators are given by

$$\hat M_{AB} = \bar\psi M_{AB}\psi, \qquad \text{where } \bar\psi = \psi^\dagger\gamma^0 = (a^\dagger_{\dot 1}, a^\dagger_{\dot 2}, b^1, b^2), \tag{2.11}$$

γ^0 and M_{AB} are four dimensional irreducible but non-unitary representation of $SO(2,4)$. They are written down in terms of four dimensional gamma matrices, the details of these are given in appendix A. These generators satisfy (2.3) and they also have the property

$$\gamma^0 M_{AB} = M^\dagger_{AB}\gamma^0, \qquad \gamma^{0\dagger} = \gamma^0. \tag{2.12}$$

These properties ensure that the generators $\hat M_{AB}$ of (2.11) which act on the Fock space of oscillators are Hermitian and they satisfy the $SO(2,4)$ algebra (2.3). This is due to the following commutation relations

$$[\hat M_{AB}, \hat M_{CD}] = \bar\psi[M_{AB}, M_{CD}]\psi. \tag{2.13}$$

which can be shown using the simple commutation rules of the oscillators (2.10). Since the generators $\hat M_{AB}$ are Hermitian, they generate an infinite dimensional but unitary representation of $SO(2,4)$ in the Fock space of oscillators $a^{\dot\gamma}, b^\gamma$.

[4] Note that we have defined the annihilation operators with raised indices.

JHEP04(2006)014

We will now discuss the properties of this Fock space. The vacuum is defined as $a^{\dot\gamma}|0\rangle = b^{\gamma}|0\rangle = 0$. In this Fock space we find it convenient to work with the generators in the maximal compact sub-group $U(1)_E \times SU(2)_L \times SU(2)_R$. As we mentioned in the previous subsection the transformation U given in (2.7) takes a representation in the maximal compact sub-group $\mathcal{H}_1$ to the usual classification of fields of the conformal group $\mathcal{H}_2$. The action of the transformation U on the Fock space is given by

$$\begin{aligned} &\hat{U} = \exp(\frac{\pi}{2}\hat{M}_{05}) = \exp(-\frac{\pi}{4}(a^\dagger b^\dagger + ba)) \\ \text{where} \qquad & a^\dagger b^\dagger = a^\dagger_{\dot 1} b^\dagger_1 + a^\dagger_{\dot 2} b^\dagger_2, \qquad ba = b^1 a^{\dot 1} + b^2 a^{\dot 2}. \end{aligned} \tag{2.14}$$

We now give the properties of $\hat{U}$ which corresponds to the ones given in (2.8)

$$\begin{aligned} &\hat{U}^{-1}\hat{K}_\mu \hat{U} = b\sigma_\mu a, \qquad \hat{U}^{-1}\hat{P}_\mu \hat{U} = -a^\dagger \bar{\sigma}_\mu b^\dagger, \\ &\hat{U}^{-1}\hat{M}_{mn}\hat{U} = \hat{M}_{mn}, \qquad \hat{U}^{-1}\hat{M}_{0m}\hat{U} = i\hat{M}_{m5}. \\ &\hat{U}^{-}(-i\hat{D})\hat{U} = \hat{E}, \\ &\qquad = \frac{1}{2}(a^\dagger a + bb^\dagger) = \frac{1}{2}(N_a + N_b) + 1. \end{aligned} \tag{2.15}$$

Here $\sigma^\mu = (-1, \sigma^m)$ and $\bar\sigma^\mu = (-1, -\sigma^m)$,

$$N_a = a^\dagger_{\dot 1} a^{\dot 1} + a^\dagger_{\dot 2} a^{\dot 2}, \qquad N_b = b^\dagger_1 b^1 + b^\dagger_2 b^2. \tag{2.16}$$

The properties given in (2.8) are most easily shown in the four dimensional representation of $SO(2,4)$ and then they are immediate in the Fock space representation because of (2.13), the details are provided in appendix B. Other useful properties of $\hat{U}$ which will be used repeatedly in the next sections is

$$U^\dagger U E = -U^\dagger U E, \qquad U^2 E = -U^2 E. \tag{2.17}$$

The above property is easily shown by repeatedly using the second line of (2.15) and $U^\dagger = U$. Finally, the left and right $SU(2)$ generators are given

$$\begin{aligned} \hat{L}_m &= \frac{1}{2}\left(\frac{1}{2}\epsilon_{mnl}\hat{M}_{nl} + \hat{M}_{m5}\right) = -a^\dagger \sigma_m a, \\ \hat{R}_m &= \frac{1}{2}\left(\frac{1}{2}\epsilon_{mnl}\hat{M}_{nl} - \hat{M}_{m5}\right) = b\sigma_m b^\dagger. \end{aligned} \tag{2.18}$$

It is now clear that why it is convenient to work in the maximal compact subgroup. From (2.15) we see that the conjugate of $\hat{K}_\mu$ are annihilation operators while that of $\hat{P}_\mu$ are creation operators. Therefore the vacuum in the Fock space satisfies

$$\hat{K}_\mu \hat{U}|0\rangle = 0. \tag{2.19}$$

The compact generator $\hat{E}$ is just the number operator in the Fock space, and the $SU(2)_L$ corresponds to the a oscillators while the $SU(2)_R$ corresponds to the b oscillators. Thus we work with the Fock space of a, b oscillators and the transform $\hat{U}$ maps to gauge invariant operators at $x = 0$.

It is important to note that the Fock space is constrained, i.e. the the number of $a^\dagger$ oscillators is equal to the number of $b^\dagger$ oscillators. This can be seen as follows: first, note that the Fock vacuum corresponds to a conformal primary because of (2.19), and the descendents are obtained from the successive actions of $\hat{P}_\mu$, which, according to (2.15), corresponds to the creation operator $a^\dagger \sigma_\mu b^\dagger$. Therefore, the states have equal number of $a^\dagger$ and $b^\dagger$. From the generators listed in (2.15) and (2.18), we see that the bilinear operators of the form $a^\dagger a^\dagger$, $b^\dagger b^\dagger$, $a^\dagger b$, and $b^\dagger a$ are missing. This constraint can be formalized by saying that the $U(1)$ given by

$$Z_1 = N_a - N_b, \tag{2.20}$$

is gauged. Therefore in the Fock space we allow only $U(1)_{Z_1}$ neutral states. Let us make a count of the generators allowed: $a^\dagger_{\dot\gamma} b^\dagger_\gamma$, $b^\gamma a^{\dot\gamma}$, $a^\dagger_{\dot\gamma} a^{\dot\delta}$, $b^\gamma b^\dagger_\delta$, these form $15+1$ generators, which form the generators of the conformal group and the central $U(1)_{Z_1}$ given by (2.20) respectively.

The $SO(6)$ R symmetry group. We construct representations of the $SO(6)$ group using eight fermionic oscillators. These are organized as 4 complex fermionic oscillators which transform in the spinor representation of $SO(6)$ as

$$\varphi = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ -\beta^{\dot 1 \dagger} \\ -\beta^{\dot 2 \dagger} \end{pmatrix}. \tag{2.21}$$

Here the fermionic oscillators obey the following anti-commutation relations

$$\{\alpha^{\tau\dagger}, \alpha_\upsilon\} = \delta^\tau_\upsilon, \quad \{\beta^{\dot\tau\dagger}, \beta_{\dot\upsilon}\} = \delta^{\dot\tau}_{\dot\upsilon}, \quad \{\alpha_\tau, \beta_{\dot\tau}\} = 0, \quad \{\alpha^{\tau\dagger}, \beta_{\dot\tau}\} = 0. \tag{2.22}$$

The Fock vacuum is defined as

$$\alpha_\tau|0\rangle = \beta_{\dot\tau}|0\rangle = 0. \tag{2.23}$$

The action of the $SO(6)$ generators in the Fock space of these oscillators is given by

$$\hat{M}_{IJ} = \varphi^\dagger M_{IJ} \varphi, \tag{2.24}$$

here M_{IJ} are four dimensional representation of generators of $SO(6)$ (more the details of these are given in appendix A). These generators satisfy the $SO(6)$ algebra given by

$$[M_{IJ}, M_{KL}] = i(\delta_{JK} M_{IL} - \delta_{IK} M_{JL} - \delta_{JL} M_{IK} + \delta_{IL} M_{JK}). \tag{2.25}$$

$I, J, K, K = 1, \ldots 6$. The fact that these generators are Hermitian ensures Hermiticity of $\hat{M}_{IJ}$. Furthermore $\hat{M}_{IJ}$ obeys the $SO(6)$ algebra since

$$[\hat{M}_{IJ}, \hat{M}_{KL}] = \varphi^\dagger [M_{IJ}, M_{KL}] \varphi. \tag{2.26}$$

$\mid$ State $\rangle$	J	E	$\hat{U}\mid$ State $\rangle$
$\lvert 0\rangle$	1	1	z
$\alpha^{\tau\dagger}\beta^{\dot{\tau}\dagger}\lvert 0\rangle$	0	1	ϕ_i; $i = 1, \dots 4$
$\alpha^{1\dagger}\alpha^{2\dagger}\beta^{\dot{1}\dagger}\beta^{\dot{2}\dagger}\lvert 0\rangle$	-1	1	$\bar{z}$

Table 1: The $SO(6)$ scalar.

It is again convenient to work in a graded decomposition of $SO(6)$ as $L^+ \oplus L^- \oplus L^0$. We now write down the generators below.

$$\begin{aligned}
&L^+ = \{\ \alpha^{\tau\dagger}\beta^{\dot{\tau}\dagger}\ \}, \qquad L^- = \{\ \alpha_\tau\beta_{\dot{\tau}}\ \} \\
&L^0 = SU(2)_{L'} \times SU(2)_{R'} \times U(1)_J, \\
&SU(2)_{L'}: \quad \alpha^\dagger\vec{\sigma}\alpha, \qquad SU(2)_{R'}: \quad \beta^\dagger\vec{\sigma}\beta, \\
&U(1)_J: \quad J = \varphi^\dagger M_{56}\varphi = \frac{1}{2}\left(2 - N_\alpha - N_\beta\right).
\end{aligned} \tag{2.27}$$

Here N_α and N_β are defined by

$$N_\alpha = \alpha^{1\dagger}\alpha_1 + \alpha^{2\dagger}\alpha_2, \qquad N_\beta = \beta^{\dot{1}\dagger}\beta_{\dot{1}} + \beta^{\dot{2}\dagger}\beta_{\dot{2}}. \tag{2.28}$$

Similarly to the case of $SO(2,4)$, the Fock space of oscillators for $SO(6)$ is constrained. It is clear from the generators listed in (2.27) that the states must have zero charge under the following $U(1)$

$$B = N_\alpha - N_\beta. \tag{2.29}$$

Starting from the Fock vacuum we can obtain all the allowed states by the repeated action of the generators in L^+. Since these generators are fermionic, the action of L^+ truncates at the second level. The table of states are given in 1.

Here we have written down the J-charge and the dimension Δ of the corresponding states. Note that the fermionic creation operators have J-charge $-1/2$. In the last column of the table 1 we have indicated the field at the origin corresponding to the state which is created by the action of $\hat{U}$. Thus these states give the scalars of $\mathcal{N} = 4$ Yang-Mills theory, which transform in the vector of $SO(6)$.

The supersymmetry generators. $\mathcal{N} = 4$ Yang-Mills admits 16 Poincaré supersymmetry generators and 16 superconformal supersymmetries. Their realization in terms of the oscillator construction is given by

The algebra of these generators realize the odd part of the $\mathcal{N} = 4$ superconformal algebra. The (anti-)commutation relations of these generators are given in appendix C. From the supersymmetry generators given in the table 2 it is clear that the complete $U(1)$ gauging is given by

$$Z = Z_1 + B = N_a - N_b + N_\alpha - N_\beta. \tag{2.30}$$

As an example, consider the generators Q^-: the number of a's is equal to the number of β's, thus respecting $U(1)_Z$ invariance.

Supersymmetry	Operator	E	J
Q^+	$a^\dagger_{\dot\gamma}\alpha_\tau$	$\frac{1}{2}$	$\frac{1}{2}$
	$b^\dagger_\gamma\beta_{\dot\tau}$	$\frac{1}{2}$	$\frac{1}{2}$
Q^-	$a^\dagger_{\dot\gamma}\beta^{\dot\tau\dagger}$	$\frac{1}{2}$	$-\frac{1}{2}$
	$b^\dagger_\gamma\alpha^{\tau\dagger}$	$\frac{1}{2}$	$-\frac{1}{2}$
S^-	$a^{\dot\gamma}\alpha^{\tau\dagger}$	$-\frac{1}{2}$	$-\frac{1}{2}$
	$b^\gamma\beta^{\dot\tau\dagger}$	$-\frac{1}{2}$	$-\frac{1}{2}$
S^+	$a^{\dot\gamma}\beta_{\dot\tau}$	$-\frac{1}{2}$	$\frac{1}{2}$
	$b^\gamma\alpha_\tau$	$-\frac{1}{2}$	$\frac{1}{2}$

Table 2: Supersymmetry generators.

2.3 Partition function

In this section we will first evaluate the partition function and show the Fock space of the oscillators together with the $U(1)_Z$ gauging is equivalent to all the on shell single letter spectrum of $\mathcal{N}=4$ Yang-Mills We then construct composite single trace operators of given length l by considering l copies of the Fock space of oscillators together with the cyclicity constraint.

Single bit partition function. The partition function for the Fock space is given by

$$\mathcal{Z} = \oint_{|x|=1} \frac{dx}{x}\mathrm{Tr}(q^{2E}x^{Z}). \tag{2.31}$$

Here the trace is over all the states in the Fock space, the contour integral around the unit circle imposes the constraint $Z=0$ on the states. It is simple to evaluate the trace as these oscillators are free, we obtain

$$\begin{aligned}\mathcal{Z} &= \oint_{|x|=1} \frac{dx}{x} q^2 \frac{(1+x)^2(1+x^{-1})^2}{(1-qx)^2(1-qx^{-1})^2}, \\ &= 2q^2\frac{(3-q)}{(1-q)^3}, \\ &= 6q^2+16q^3+30q^4+48q^5+70q^6+\cdots .\end{aligned} \tag{2.32}$$

Note that this single letter partition function agrees with the on shell single letter spectrum of $\mathcal{N}=4$ YM at $\lambda=0$ evaluated in [33−35, 13, 14]

We now give an account of the numbers occurring for the first few terms in the expansion of the partition function in powers of q. From (2.32) we note there are 6 states at $E=1$, these are due to the 6 scalars ϕ^I in the fundamental of $SO(6)$. At $E=3/2$ we see that there are 16 states, these are accounted for by the 16 fermions of $\mathcal{N}=4$ Yang-Mills, which can be thought of as the 16 component Weyl-Majorana fermion in ten dimensions, ψ. For $E=2$, we have the 24 states from $\partial_\mu\phi^I$ [5] and 6 states from $F_{\mu\nu}$. Together they give rise to the term $30q^4$ in the partition function. At the next level $E=5/2$, we have the

[5] At $\lambda=0$, the covariant derivative reduces to the ordinary derivative.

JHEP04(2006)014

State	$\hat{U}\vert\text{State}\rangle$
$\vert 0\rangle,\quad \alpha^{\tau\dagger}\beta^{\dot\tau\dagger}\vert 0\rangle,\quad \alpha^{1\dagger}\alpha^{2\dagger}\beta^{\dot 1\dagger}\beta^{\dot 2\dagger}\vert 0\rangle$	ϕ^I
$a^\dagger_{\dot\gamma}\beta^{\dot\tau\dagger}\vert 0\rangle,\quad b^\dagger_\gamma\alpha^{\tau\dagger}\vert 0\rangle,\quad a^\dagger_{\dot\gamma}\beta^{\dot 1\dagger}\beta^{\dot 2\dagger}\alpha^{\tau\dagger}\vert 0\rangle,\quad b^\dagger_\gamma\alpha^{1\dagger}\alpha^{2\dagger}\beta^{\dot\tau\dagger}\vert 0\rangle$	ψ
$a^\dagger_{\dot\gamma}a^\dagger_{\dot\delta}\beta^{\dot 1\dagger}\beta^{\dot 2\dagger}\vert 0\rangle,\quad b^\dagger_\gamma b^\dagger_\tau\alpha^{1\dagger}\alpha^{2\dagger}\vert 0\rangle$	$F_{\mu\nu}$

Table 3: Basic letters of $\mathcal{N}=4$ YM and oscillator states.

states of $\partial_\mu\psi$, they give rise to $4\times 16 = 64$ states, but on examining (2.32) we see that there are only 48 states. This is because the states of the Fock space are in correspondence with the on shell single letters of the Yang-Mills, indeed on subtracting the 16 components of the equations of motion $\gamma\cdot\partial\psi = 0$ from 64 we obtain the 48 on shell states. Let us consider one more level to illustrate the fact the Fock space corresponds to on shell states. At $E=3$ there are the following states $\partial_\mu\partial_\nu\phi^I$ and $\partial_\mu F_{\nu\rho}$, their number are $60+24$ respectively. 6 states out of the 60 vanish due to the on shell condition $\partial^2\phi^I = 0$. 8 states out the 24, correspond to $\partial^\mu F_{\mu\nu} = 0$ and Bianchi identity $\partial_{[\mu}F_{\nu\sigma]} = 0$. Subtracting these on shell degrees we get $(60+24)-(6+4+4) = 70$, which agrees with the partition function. Below we list the basic Fock space states and the corresponding fields.

We now examine the reason why the Fock space includes only the on shell letters of $\mathcal{N}=4$ Y-M. Consider the operator $\partial^\mu\partial_\mu$. From (2.15) we see that on the Fock space the operator is represented by $a^\dagger\bar\sigma^\mu b^\dagger a^\dagger\bar\sigma_\mu b^\dagger$. One can easily see, using the identity [36]

$$\bar\sigma^{\mu\dot\gamma\gamma}\bar\sigma_\mu^{\dot\tau\tau} = -2\epsilon^{\dot\gamma\dot\tau}\epsilon^{\gamma\tau}, \tag{2.33}$$

and the fact that the ϵ-tensor is anti-symmetric in its indices, that the representation of ∂^2 on the Fock space vanishes. Similarly, consider the equation of motion of the fermion $a^\dagger_{\dot\gamma}\beta^{\dot\tau\dagger}|0\rangle$. This is given by

$$\bar\sigma^{\mu\dot\gamma\gamma}P_\mu a^\dagger_{\dot\gamma}\beta^{\dot\tau\dagger}|0\rangle = \bar\sigma^{\mu\dot\gamma\gamma}(a^\dagger\bar\sigma_\mu b^\dagger)a^\dagger_{\dot\gamma}\beta^{\dot\tau\dagger}|0\rangle = 0, \tag{2.34}$$

[6]. where we have again applied (2.33). Using the same method and (2.33) one can show that the equation of motion of the other fermions and $\partial^\mu F_{\mu\nu} = 0$ is identically satisfied in the oscillator construction. To show that the Bianchi identity $\partial_{[\mu}F_{\nu\rho]} = 0$ is satisfied, consider the anti-self dual component $a^\dagger_{\dot\gamma}a^\dagger_{\dot\delta}\beta^{\dot 1\dagger}\beta^{\dot 2\dagger}|0\rangle$. The Bianchi identity can be written as

$$\epsilon^{\lambda\mu\nu\rho}(a^\dagger\sigma_\mu b^\dagger)(a^\dagger\bar\sigma_{\nu\rho}a^\dagger)\beta^{\dot 1\dagger}\beta^{\dot 2\dagger}|0\rangle = 2i(a^\dagger\sigma_\mu b^\dagger)(a^\dagger\bar\sigma^{\lambda\mu}a^\dagger)|0\rangle = 0. \tag{2.35}$$

To show the above term vanishes, one again uses (2.33) after expressing $\bar\sigma^{\lambda\mu}$ in terms of $\bar\sigma$ and σ. Similarly, one can show that the Bianchi identity is satisfied by the anti-self dual component of the field strength. Using the fact that the basic letters are on shell it is easy to see that derivatives of these letters will also be on shell.

Multi-bit partition function. Once we have obtained all the letters of $\mathcal{N}=4$ Yang-Mills, it is easy to construct the gauge invariant words using the oscillators. Consider a

[6]From now on we use the same symbol for P_μ and its conjugate $(a^\dagger\bar\sigma b^\dagger)$

single trace gauge invariant operator of length l. To construct the oscillator representation of this operator one examines the Fock space $\otimes_{s=1}^{l}\mathcal{H}_s$, where $\mathcal{H}$ refers to the Fock space of a single copy of the oscillators a, b, α, β. The state at site s of the Fock space is constructed such that it corresponds to the operator at the site s of the gauge invariant word. Then one sums over all cyclic permutations so that the oscillator state is invariant under the cyclic shift of the sites. Each term in the sum is weighted by the sign of the permutation, the sign of the permutation is minus if the total number of exchanges involving a fermion is odd. This sum over cyclic permutation is performed since a single trace operator is invariant under this weighted cyclic permutation. Symbolically the Hilbert space of single trace operator is denoted by

$$\mathcal{H}^{(l)} = \sum_{\pi} \mathrm{sign}(\pi) \otimes_{s=1}^{l} \mathcal{H}, \tag{2.36}$$

where π refers to a cyclic permutation of length l and $\mathrm{sign}(\pi) = -1$ if the number of exchanges involving a fermion is odd, else $\mathrm{sign}(\pi) = 1$. To illustrate this construction, we provide the following simple example of the operator $\mathrm{Tr}(\phi^i Z^{l-1})$,

$$\sum_{\pi} \prod_{s=1}^{l} \left(O(x)^{\pi_s} |0\rangle^{(s)} \right) \leftrightarrow \frac{1}{\sqrt{N^l}} \mathrm{Tr}(\phi^i Z^{l-1}). \tag{2.37}$$

Here[7]

$$\begin{aligned} O(x)^{(1)} &= \frac{1}{\sqrt{2}} \exp(ixP) U(\alpha^\dagger \sigma^i \beta^\dagger)|0\rangle, \\ O(x)^{(2)} &= O(x)^{(3)} \ldots = \exp(ixP) U|0\rangle. \end{aligned} \tag{2.38}$$

It is clear from the method of construction of the single trace operators, that, if one evaluates the multi-bit partition function, it will agree with that of the single trace operators. To show this explicity we will evaluate the multi-bit partition function. In doing this, in order to keep the discussion simple we will not subtract the contributions of states that can be written as $\mathrm{Tr}(\mathrm{Fermion})^2$, which vanish by Fermi statistics. Let us define g as the generator of the cyclic group, *i.e.*

$$g|s_1 > |s_2 > ...|s_n >= |s_2 > ...|s_n > |s_1 > . \tag{2.39}$$

Inserting g^t into the partition function of l bits we obtain

$$\oint \prod_{s=1}^{l} \frac{dx_s}{x_s} \mathrm{Tr}(q^{2\sum_{s=1}^{l} E(s)} \prod x_s^{Z(s)} g^t) = \left(\mathcal{Z}(q^{\frac{l}{(t,l)}} \right). \tag{2.40}$$

here $E(s), Z(s)$ refer to the operators E, Z at site s, and the integrations over x_s implement the local $U(1)$ constraint. (l,t) denotes the largest common divisor of t and l. To project

[7] For the remaining part of the paper we will drop the hat, $\hat{}$ in the Fock space representation of the generators of $SO(2,6)$.

on to the cyclically invariant states we insert the projector $P = \frac{1}{l}(1 + g + g^2 + ... + g^{l-1})$. into the multi-bit partition function. Using (2.40) we obtain

$$\mathcal{Z}_l = \frac{1}{l}\sum_{s=1}^{l}\left(\mathcal{Z}(q_2^{\frac{l}{(s,l)}})\right)^{(s,l)},$$
$$= \sum_{d|l}\frac{\varphi(d)}{l}\mathcal{Z}(q^d)^{\frac{l}{d}}. \tag{2.41}$$

In the second line of the above equation we have re-arranged the summation over s to the sum over the divisors of n. Then, $\varphi(d)$ denotes the number of s, such that $(s,l) = \frac{l}{d}$. Since $(s,l) = \frac{n}{d}$, this implies $s = \frac{l}{d}t$ with $(t,d) = 1$. On the other hand we also have $s \leq l$ then $t < d$ (unless $d = 1$) so $\varphi(d)$ is given by the number of co-primes with d and smaller than d, with $\varphi(1) = 1$. But this is the definition of Euler's totient function. To compare with the partition function of all the single trace operators of $\mathcal{N} = 4$ Yang-Mills we sum over all the lengths l from 2 to ∞, we neglect the case of a single bit. This gives

$$\mathcal{Z}_{\text{Singletrace}} = \sum_{l=2}^{\infty}\sum_{d|l}\frac{\varphi(d)}{l}\mathcal{Z}_{\Delta}(q^d)^{\frac{l}{d}} = 21q^4 + 96q^5 + 392q^6 + 1344q^7 + ..., \tag{2.42}$$

Note that we have also included the contributions of states of the type $\mathrm{Tr}(\mathrm{Fermion})^2$, which can be subtracted easily. It is clear that using the oscillators it is also easy to evaluate partition functions which carry information of $SO(6)$ or $SO(2,4)$ quantum numbers as done in [13, 14].

3. String overlap at $\lambda = 0$

In the previous section we have seen that gauge invariant operators of, say, length l, can be represented as a state in the Hilbert space $\mathcal{H}^{(l)}$. We refer to such a state in the Hilbert space as a string. The world sheet of such a string is discrete and composed of l bits. In this section we show that the planar two point and three point functions of gauge invariant operators at $\lambda = 0$ can be reproduced just by geometric overlap rules of their corresponding states in the Hilbert space. We also write the geometric overlaps in terms of 2-string vertex and a 3-string vertex for the two point and 3-point functions respectively.

3.1 Single bit overlap

To show that the inner product of two string states in the Hilbert space $\mathcal{H}^{(l)}$ reduces to the two point function of the corresponding gauge invariant operator we first study the overlap of single bit states. This is done using three methods: the first method is a direct evaluation of the inner product using just the oscillator algebra; in the second approach we show that the inner product satisfies conformal ward identities which, this enables us to determine the overlap; finally we use the geometric meaning of the operators involved in the overlap to evaluate it.

Direct evaluation of inner product. Consider the vacuum state at position x, which is given by $e^{ixP}U|0\rangle$. We have shown in the previous section that this corresponds to the field $z(x)$. The inner product of this state with the another vacuum state at position y is given by

$$I(x,y) = \langle 0|U^\dagger e^{-i(x-y)P}U|0\rangle. \tag{3.1}$$

Using the definition of U in (2.14) and the formulae in (2.15) the above expression can be written as

$$\begin{aligned} I(x,y) &= \langle 0|U^2 \exp\left[i(a^\dagger\bar\sigma b^\dagger)\cdot(y-x)\right]|0\rangle, \\ \text{with} \quad U^2 &= \exp\left[-\frac{\pi}{2}(a^\dagger b^\dagger + ba)\right]. \end{aligned} \tag{3.2}$$

To evaluate the above inner product we use the identity

$$\begin{aligned} U_t^2 &= \exp t(a^\dagger b^\dagger + ba), \\ &= \exp\left(a^\dagger b^\dagger \tan t\right)\exp\left[-(a^\dagger a + bb^\dagger)\ln\cos t\right]\exp\left(ba\tan t\right). \end{aligned} \tag{3.3}$$

This identity is shown as follows: first differentiate both sides of the above equation with respect to t. Then move all the factors to the extreme left. This, together with the condition $U_0^2 = 1$, results in the above identity. Substituting (3.3) in (3.2) we obtain

$$\begin{aligned} I(x,y) &= \lim_{t\to-\pi/2}\frac{1}{\cos^2 t}\frac{1}{1-2(x^0-y^0)\tan t + (x-y)^2\tan^2 t}, \\ &= \frac{1}{(x-y)^2}. \end{aligned} \tag{3.4}$$

Here we have also used the following property of squeezed states,

$$\langle 0|\exp\left(\frac{1}{2}a\cdot M\cdot a\right)\exp\left(\frac{1}{2}a^\dagger\cdot N\cdot a^\dagger\right)|0\rangle = \left[\mathrm{Det}(1-M\cdot N)\right]^{-1/2}, \tag{3.5}$$

where a_i's refer to n oscillators, M and N are $n\times n$ matrices. From (3.4) we see that the overlap of a single bit corresponding to the operator $z(x)$ is identical to the two point function of $\langle\bar z(x)z(y)\rangle$. It is worth noting that the transformation U is not a unitary transformation, in fact the norm of the state at a given position x is infinite, where as the norm of the state in the Hilbert space $\mathcal{H}_1$ is finite.

Method of conformal Ward identity. Another method to show that the overlap of the vacuum state reduces to the two point function of the scalar $z(x_1)$, is to show that the overlap satisfies the conformal Ward identities. This method does not rely on the direct manipulation of the oscillators but on the properties of the generators of the conformal group. Using the properties of the operator U given in (2.15), the two point function $I(x,y)$ can also be written as

$$I(x,y) = \langle 0|\exp(i(b\bar\sigma^\dagger a)\cdot y)U^\dagger U\exp(-i(a^\dagger\bar\sigma b^\dagger\cdot x)|0\rangle. \tag{3.6}$$

We first consider the inner product,

$$\langle 0|\exp(i(b\bar\sigma^\dagger a)\cdot y)U^\dagger U E\exp(-i(a^\dagger\bar\sigma b^\dagger)\cdot x)|0\rangle = (x\partial^x + 1)I(x,y), \tag{3.7}$$

JHEP04(2006)014

where we have used the relation

$$[E, \exp(-i(a^\dagger \bar{\sigma} b^\dagger) \cdot x)] = x\partial^x \exp(-i(a^\dagger \bar{\sigma} b^\dagger) \cdot x), \tag{3.8}$$

to write the insertion of E as a differential operator on the overlap. The shift of 1 in the operator is due to the normal ordering constant in E. We now can use the identity (2.17), $U^\dagger U E = -E U^\dagger U$ to move the operator E to the left. Thus we obtain

$$\langle 0| \exp(i(b\bar{\sigma}^\dagger a) \cdot y) E U^\dagger U \exp(-i a^\dagger \bar{\sigma} b^\dagger) \cdot x)|0\rangle = -(y\partial^y + 1) I(x, y), \tag{3.9}$$

here again we have converted the action of E as a differential operator on the bra state. Now comparing the equations (3.7) and (3.9) we obtain

$$(x\partial^x + y\partial^y + 2) I(x, y) = 0. \tag{3.10}$$

The above equation is the conformal Ward identity for a primary field of weight 1. Using a similar procedure we can show that the overlap also satisfies translational invariance. Inserting the operator $a^\dagger \bar{\sigma}_\mu b^\dagger$ which is conjugate to the momentum operator (see (2.15)) in (3.6) we obtain

$$\langle 0| \exp(i(b\bar{\sigma}^\dagger a) \cdot y) U^\dagger U (a^\dagger \bar{\sigma}_\mu b^\dagger) \exp(-i(a^\dagger \bar{\sigma} b^\dagger) \cdot x)|0\rangle = i\partial^x_\mu I(x, y). \tag{3.11}$$

Here we have converted insertion of momentum operator to a derivative. Using $U^\dagger U(a^\dagger \bar{\sigma}_\mu b^\dagger) = (b\sigma^\dagger_\mu a) U^\dagger U$, which can be obtained from (2.15), we can move momentum operator to the left and then again convert it to a derivative, this results in

$$\langle 0| \exp(i(b\bar{\sigma}^\dagger a) \cdot y)(b\bar{\sigma}^\dagger_\mu a) U^\dagger U \exp(-i(a^\dagger \bar{\sigma} b^\dagger) \cdot x)|0\rangle = -i\partial^y_\mu I(x, y). \tag{3.12}$$

Again comparing (3.11) and (3.12) we obtain the momentum conservation equation.

$$(\partial^x_\mu + \partial^y_\mu) I(x, y) = 0 \tag{3.13}$$

Using (3.13) we can replace the derivative with respect to y in (3.10) to get the equation

$$[(x - y)\partial_x + 2]\, I(x, y) = 0, \tag{3.14}$$

the solution of this equation is the required two point function $I(x, y) = 1/(x - y)^2$.

The geometric method. It is instructive to illustrate yet another method to show that the overlap of the vacuum state is the two point function of the scalar $z(x)$. This method relies on expressing the action of the oscillators as differential operators, this method is convenient when one wants to obtain the two point functions or three points functions at one-loop. Consider the overlap again

$$I(x, y) = \langle 0| \exp(i(b\bar{\sigma}^\dagger a) \cdot y) U^\dagger U |x\rangle, \tag{3.15}$$

here $|x\rangle$ refers to the state $\exp(-i(a^\dagger\bar\sigma b^\dagger)\cdot x)|0\rangle$. Now we convert each of the operators in (3.15) from oscillators to differential operators acting on the state $|x\rangle$ from right to left. The action of $U^\dagger U$ on the state $|x\rangle$ can be written in terms of the differential operator

$$\begin{aligned} U^\dagger U|x\rangle &= \exp\left(\frac{\pi}{2}(P_0-K_0)\right)|x\rangle, \\ &= \exp\left[-\frac{i\pi}{2}\left(\partial_0^x + 2x_0 + 2x_0 x\cdot\partial^x - x^2\partial_0^x\right)\right]|x\rangle. \end{aligned} \tag{3.16}$$

Here we have used the fact that

$$\begin{aligned} P_\mu|x\rangle &= -(a^\dagger\bar\sigma_\mu b^\dagger)\exp(-i(a^\dagger\bar\sigma b^\dagger)\cdot x)|0\rangle, \\ &= -i\partial_\mu^x|x\rangle = P_\mu^{(x)}|x\rangle, \\ K_\mu|x\rangle &= (b\sigma_\mu a)\exp(-i(a^\dagger\bar\sigma b^\dagger)\cdot x)|0\rangle, \\ &= i(2x_\mu + 2x_\mu x\cdot\partial^x - x^2\partial_\mu^x)|x\rangle, \\ &= K_\mu^{(x)}|x\rangle. \end{aligned} \tag{3.17}$$

Now one can convert the operator $\exp(i(b\bar\sigma^\dagger a)\cdot y)$ as a differential operator acting on the state $|x\rangle$, we then obtain

$$I(x,y) = \langle 0|\exp\left(\frac{\pi}{2}P_0^{(x)} - K_0^{(x)})\right)\exp(\tilde y\cdot K^{(x)})|x\rangle. \tag{3.18}$$

Here $\tilde y = (y^0, -y^m)$, this is due to the fact the $K_\mu^{(x)}$ corresponds to the operator $(b\sigma_\mu a)$, while we have the operator $(b\bar\sigma_\mu^\dagger a)$. Note that in the process of converting the oscillators to differential operators there is a reversal in the order of action. The next step is to explictly perform the action of the operators on the state $|x\rangle$. This gives

$$\begin{aligned} I(x,y) &= \langle 0|\exp\left(\frac{\pi}{2}(K_0^{(x)} - P_0^{(x)})\right)\frac{1}{1+2\tilde y\cdot x + \tilde y^2 x^2}|x'\rangle, \\ \text{where}\quad x'^\mu &= \frac{x^\mu + \tilde y^\mu}{1+2\tilde y\cdot x + y^2x^2}, \end{aligned} \tag{3.19}$$

here we have used the action of finite special conformal transformation $K_\mu^{(x)}$, by an amount $\tilde y$ on a scalar primary of weight 1. To perform the action of the operation $\exp(\frac{\pi}{2}(P_0^{(x)} - K_0^{(x)}))$ one notes that it is the rotation $\exp(\pi M_{05})$, and it acts on the coordinates as [8]

$$x^0 \to -\frac{x^0}{x^2} \qquad x^m \to \frac{x^m}{x^2}, \tag{3.20}$$

performing this operation on (3.19), we obtain

$$\begin{aligned} I(x,y) &= \frac{x^2}{(x-y)^2}\frac{1}{x^2}\langle 0|x''\rangle, \qquad x''^0 = \frac{-x^0+y^0}{(x-y)^2}, \quad x''^m = \frac{x^m - y^m}{(x-y)^2}, \\ &= \frac{1}{(x-y)^2}. \end{aligned} \tag{3.21}$$

In the above equation we have used $\langle 0|x''\rangle = 1$ which is easy to see from the definition of $|x''\rangle$. Thus again we have obtained the required two point function of the scalar $\langle \bar z(x)z(y)\rangle$.

[8] $\exp(\pi M_{05})$ acts linearly on the coordinates η^A introduced in [37], then one can restrict its action on the light cone to realize its transformation on the coordinate $x^\mu = \eta^\mu/(\eta^5+\eta^6)$.

Single bit overlap for $SO(6)$ scalars. It is now easy to see that the two point functions of the $SO(6)$ scalars also are given by inner product of the corresponding bit states. This is because the other scalars are obtained by the action of creation operators $\alpha^{\tau\dagger}\beta^{\dot{\tau}\dagger}$ on the vacuum state, these oscillators commute with U^2 and the translation operators $\exp(ixP)$. The inner product factorizes as the inner product of the bosonic oscillators and the inner product of fermionic oscillators. Thus the position dependence of the overlap is entirely governed by the bosonic oscillators $\{a, b\}$. The overlap of the fermionic oscillators ensure that the Krönecker delta in the two point function $\langle\phi^I(x)\phi^J(y)\rangle = \delta_{IJ}/(x-y)^2$ is reproduced.

Single bit overlap for fermions. Let us now examine the two point function of the fermions. Consider the state $\exp(ixP)Ua^\dagger_{\dot{\gamma}}\beta^{\dot{\tau}\dagger}|0\rangle$ which can also be written as $Ua^\dagger_{\dot{\gamma}}\beta^{\dot{\tau}\dagger} \times \exp(-i(a^\dagger\bar{\sigma}b^\dagger)\cdot x)|0\rangle$. The overlap of this state with the same state at position y and with indices $\dot{\delta}, \dot{\upsilon}$ is given by

$$
\begin{aligned}
F(x,y) &= \delta^{\dot{\tau}}_{\dot{\upsilon}}\langle 0|\exp(i(b\bar{\sigma}^\dagger a)\cdot y)a^{\dot{\delta}}U^\dagger U a^\dagger_{\dot{\gamma}}\exp(-i(a^\dagger\bar{\sigma}b^\dagger)\cdot x)|0\rangle, \\
&= \delta^{\dot{\tau}}_{\dot{\upsilon}}\langle 0|\exp(i(b\bar{\sigma}^\dagger a)\cdot y)U^\dagger U a^\dagger_{\dot{\gamma}}b^\dagger_{\dot{\delta}}\exp(-i(a^\dagger\bar{\sigma}b^\dagger)\cdot x)|0\rangle.
\end{aligned} \tag{3.22}
$$

The Krönecker delta is due to the inner product of the fermions, in the second line we have moved $a^{\dot{\delta}}$ to the right of $U^\dagger U$ using the following equation

$$
a^{\dot{\delta}}U^\dagger U = U^\dagger U b^\dagger_{\dot{\delta}}\,. \tag{3.23}
$$

The above identity can be shown using the from the basic identity in (3.3) and then commuting $a^{\dot{\delta}}$ to the right. Rewriting the combination $a^\dagger_{\dot{\gamma}}b^\dagger_{\dot{\delta}}$ as a derivative we obtain

$$
\begin{aligned}
F(x,y) &= -\frac{i}{2}\delta^{\dot{\tau}}_{\dot{\upsilon}}\sigma_{\dot{\delta}\dot{\gamma}}\cdot\partial^x\langle 0|\exp(i(b\bar{\sigma}^\dagger a)\cdot y)U^\dagger U\exp(-i(a^\dagger\bar{\sigma}b^\dagger)\cdot x)|0\rangle, \\
&= -\frac{i}{2}\delta^{\dot{\tau}}_{\dot{\upsilon}}\sigma_{\dot{\delta}\dot{\gamma}}\cdot\partial^x\frac{1}{(x-y)^2}.
\end{aligned} \tag{3.24}
$$

The last line has the required correlation function of fermions. It is clear using the above manipulation the overlap of all the oscillators corresponding to fermions in table 3, reduces to the required two point function.

Single bit overlap for field strength. Finally consider the two point function of bits corresponding to the field strength $F_{\mu\nu}$. From table 3. we see that this state is given by

$$
F_{\mu\nu}(x)|0\rangle = \exp(ixP)U\left((a^\dagger\bar{\sigma}_{\mu\nu}a^\dagger)\beta^{\dot{1}\dagger}\beta^{\dot{2}\dagger}|0\rangle + (b^\dagger\sigma_{\mu\nu}b^\dagger)\alpha^{1\dagger}\alpha^{2\dagger}\right)|0\rangle, \tag{3.25}
$$

here we have written the field strength as the sum of the self-dual and anti-self-dual components. The overlap of this state at position x and y is given by

$$
\begin{aligned}
G_{\mu\nu;\rho\sigma}(x,y) &= \langle 0|F_{\mu\nu}(y)F_{\rho\sigma}(x)|0\rangle, \\
&= -\langle y|U^\dagger U(a^\dagger\bar{\sigma}_{\mu\nu}a^\dagger)(b^\dagger\sigma_{\rho\sigma}b^\dagger)|x\rangle - ((\mu,\nu)\leftrightarrow(\rho,\sigma))\,.
\end{aligned} \tag{3.26}
$$

In obtaining the above equation we have repeatedly used the identity (3.23) and $\bar{\sigma}^{\dagger}_{\mu\nu} = -\sigma_{\mu\nu}$. Now rewriting the pairs $a^{\dagger}_{\dot{\gamma}} b^{\dagger}_{\gamma}$ as derivatives on the state $|x\rangle$, we obtain

$$G_{\mu\nu;\rho\sigma}(x,y) = \frac{1}{16}\bar{\sigma}^{\dot{\gamma}\dot{\delta}}_{\mu\nu}\sigma^{\gamma\delta}_{\rho\sigma}\left(\sigma^{\lambda}_{\gamma\dot{\gamma}}\sigma^{\varrho}_{\delta\dot{\delta}} + \sigma^{\varrho}_{\gamma\dot{\gamma}}\sigma^{\lambda}_{\delta\dot{\delta}} + \sigma^{\lambda}_{\delta\dot{\gamma}}\sigma^{\varrho}_{\gamma\dot{\delta}} + \sigma^{\varrho}_{\delta\dot{\gamma}}\sigma^{\lambda}_{\gamma\dot{\delta}}\right)\partial_\lambda\partial_\varrho I(x,y) + ((\mu,\nu)\leftrightarrow(\rho\sigma)). \tag{3.27}$$

We now substitute the following identity [36] in the above equation

$$\sigma^{\mu}_{\gamma\dot{\gamma}}\sigma^{\nu}_{\delta\dot{\delta}} + \sigma^{\nu}_{\gamma\dot{\gamma}}\sigma^{\mu}_{\delta\dot{\delta}} = -\eta^{\mu\nu}\epsilon_{\gamma\delta}\epsilon_{\dot{\gamma}\dot{\delta}} + 4\sigma^{\rho\mu}_{\alpha\beta}\bar{\sigma}^{\rho\nu}_{\dot{\alpha}\dot{\beta}}, \tag{3.28}$$

to obtain

$$G_{\mu\nu;\rho\sigma}(x,y) = \frac{1}{2}\left[\mathrm{Tr}(\sigma^{\rho\sigma}\sigma^{\kappa\varrho})\mathrm{Tr}(\bar{\sigma}^{\mu\nu}\bar{\sigma}^{\kappa\lambda}) + ((\mu,\nu)\leftrightarrow(\rho,\sigma))\right]\partial_\lambda\partial_\varrho I(x,y). \tag{3.29}$$

To further simplify the above equation we need the following identities [36].

$$\begin{aligned}\mathrm{Tr}(\sigma^{\rho\sigma}\sigma^{\kappa\varrho}) &= \frac{1}{2}\left(\eta^{\rho\varrho}\eta^{\sigma\kappa} - \eta^{\rho\kappa}\eta^{\sigma\varrho} - i\epsilon^{\rho\sigma\kappa\varrho}\right), \\ \mathrm{Tr}(\bar{\sigma}^{\rho\sigma}\bar{\sigma}^{\kappa\varrho}) &= \frac{1}{2}\left(\eta^{\rho\varrho}\eta^{\sigma\kappa} - \eta^{\rho\kappa}\eta^{\sigma\varrho} + i\epsilon^{\rho\sigma\kappa\varrho}\right).\end{aligned} \tag{3.30}$$

Substituting the above identities in the last line of (3.27) we obtain

$$\begin{aligned}G_{\mu\nu;\rho\sigma}(x,y) &= \frac{1}{2}\Big[(\eta^{\rho\varrho}\eta^{\mu\lambda}\eta^{\sigma\nu} - \eta^{\rho\varrho}\eta^{\sigma\mu}\eta^{\nu\lambda} - \eta^{\sigma\varrho}\eta^{\mu\lambda}\eta^{\rho\nu} + \eta^{\rho\mu}\eta^{\sigma\varrho}\eta^{\nu\lambda} \\ &\qquad - \frac{1}{2}(\eta^{\rho\mu}\eta^{\sigma\nu}\eta^{\varrho\lambda} - \eta^{\rho\nu}\eta^{\sigma\mu}\eta^{\varrho\lambda})\Big]\partial_\lambda\partial_\varrho I(x,y) \\ &= \frac{1}{2}\left(\eta^{\rho\varrho}\eta^{\mu\lambda}\eta^{\sigma\nu} - \eta^{\rho\varrho}\eta^{\sigma\mu}\eta^{\nu\lambda} - \eta^{\sigma\varrho}\eta^{\mu\lambda}\eta^{\rho\nu} + \eta^{\rho\mu}\eta^{\sigma\varrho}\eta^{\nu\lambda}\right)\partial_\lambda\partial_\varrho I(x,y).\end{aligned} \tag{3.31}$$

Notice that the second term in the square brackets just implements the tracelessness condition in the indices λ, ϱ. The second line in the above equation is written with the understanding that the on shell condition, that is $\Box I(x,y) = 0$ implements the tracelessness condition in λ, ϱ. (3.31) is required two point function for the gauge field strength.

3.2 Two-point functions

In the previous subsection we have shown that the inner product of all single bit states reduces to the two point function of the corresponding fields. A gauge invariant operator of length l corresponds to a state in the Hilbert space $\mathcal{H}^{(l)}$. Inner product in this space is simply the inner product inherited from the inner product of the single bit states. To be more explicit, consider a state made of the product of single bit operators $\hat{O}^{(1)}\hat{O}^{(2)}\cdots\hat{O}^{(l)}$, then the corresponding state in the Hilbert space $\mathcal{H}^{(l)}$ is given by

$$\hat{\mathcal{O}}(x)|0\rangle = \sum_{\pi}\mathrm{sign}(\pi)\prod_{i=1}^{l}\left(\hat{O}^{(\pi_i)}(x)|0\rangle^{(i)}\right), \tag{3.32}$$

here the summation runs over all the cyclic permutations π of the set of integers $\{1,2,\ldots l\}$. The state $|0\rangle^{(i)}$, refers to the i-th bit vacuum, while the vacuum $|0\rangle$ on the left hand side of the above equation just refers to the product of these single bit vacua. The operator $O^{(\pi_i)}$

stands for the $O^{\pi(i)}$-th operator, but written in terms of the oscillators of the the i-th Fock space. The $\mathrm{sign}(\pi)$ just refers to the total sign one obtains after the cyclic permutations of the operator $\hat{O}^{(1)}\hat{O}^{(2)}\cdots\hat{O}^{(l)}$. If in the cyclic permutation two fermionic single bit operators are exchanged, there is negative sign. Bosonic single bit operators are exchanged without any contribution to the sign. The sum of cyclic permutations with the sign as the weight essentially implements the cyclic symmetry of the trace in the string bit language. The inner product of two such states is defined by the following.

$$\langle 0|\hat{\mathcal{O}}^\dagger(x)\hat{\mathcal{O}}(y)|0\rangle = \frac{1}{l}\sum_{\pi\sigma}\mathrm{sign}(\pi)\mathrm{sign}(\sigma)\prod_{i=1}^{l}{}^{(i)}\langle 0|O^{\pi_i\dagger}(x)O^{\sigma_i}(y)|0\rangle^{(i)}. \tag{3.33}$$

The above formula defines the inner product in the Hilbert space $\mathcal{H}^{(l)}$ in terms of the overlap of the single bit Hilbert space, note that we have normalized the inner product by $1/l$ to avoid over counting of the identical overlaps. We will denote this overlaps as the 2-string overlap. This overlap can also be written as a 2-string vertex which is a state in the tensor product of the 2-string Hilbert space, this is given by

$$\begin{aligned}|V_2\rangle = \frac{1}{l}\exp\sum_{s=1}^{l}\Big(&a^{\dagger(1)}(s)a^{\dagger(2)}(s) + b^{\dagger(1)}(s)b^{\dagger(2)}(s)\\ &+\alpha^{\dagger(1)}(s)\alpha^{\dagger(2)}(s) + \beta^{\dagger(1)}(s)\beta^{\dagger(2)}(s)\Big)\,|0\rangle_{(1)}\otimes|0\rangle_{(2)}.\end{aligned} \tag{3.34}$$

Here (1) and (2) refers to the Hilbert space of the two strings, s stands for the sites on the bit strings and

$$\begin{aligned}a^{\dagger(1)}a^{\dagger(2)} &= a_{\dot 1}^{\dagger(1)}a_{\dot 1}^{\dagger(2)} + a_{\dot 2}^{\dagger(1)}a_{\dot 2}^{\dagger(2)},\\ b^{\dagger(1)}b^{\dagger(2)} &= b_1^{\dagger(1)}b_1^{\dagger(2)} + b_2^{\dot 2\dagger(1)}b_2^{\dot 2\dagger(2)},\\ \alpha^{\dagger(1)}\alpha^{\dagger(2)} &= \alpha^{1\dagger(1)}\alpha^{1\dagger(2)} + \alpha^{2\dagger(1)}\alpha^{2\dagger(2)},\\ \beta^{\dagger(1)}\beta^{\dagger(2)} &= \beta^{\dot 1\dagger(1)}\beta^{\dot 1\dagger(2)} + \beta^{2\dagger(1)}\beta^{\dot 2\dagger(2)}.\end{aligned} \tag{3.35}$$

It is clear from the structure of the 2-string vertex, it implements the delta function overlap $\delta(X^{(1)} - X^{(2)})$ where X refers to the string bit world sheet. Using the the 2-sting vertex, the 2-string overlap of operators (3.33) can be written as

$$\langle 0|\hat{\mathcal{O}}^\dagger(x)\otimes\langle 0|\hat{\mathcal{O}}^\dagger(y)|V_2\rangle. \tag{3.36}$$

It is easy to see from the definition of the 2-string vertex (3.34), the above formula reduces to (3.33).

We now show that this definition of the overlap is identical to the rules of planar Wick contractions one performs in evaluating the two point function in the gauge theory. The comparison of this overlap with planar Wick contractions performed in obtaining the two point function of two single trace operators is show in the figure (1). The horizontal lines in the bit string overlap denotes the single bit inner product. In the previous subsection we have shown that single bit inner product reduces to the two point function of the corresponding single letters of the Yang-Mills. According to the definition in (3.33) we

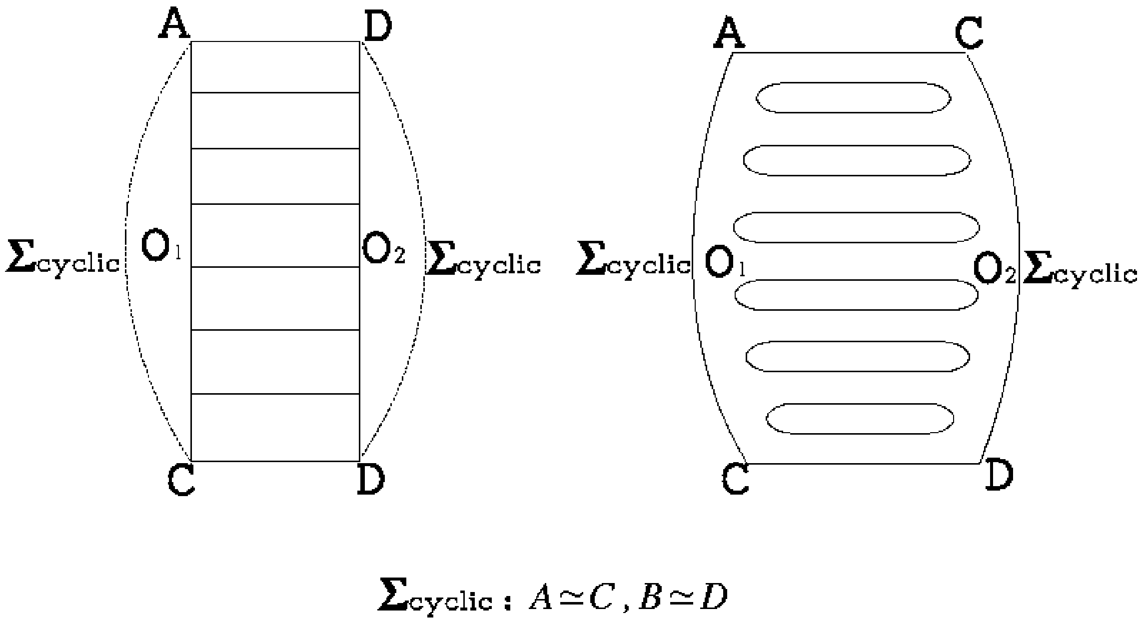

Figure 1: Bit overlap and two point function

have to sum over the cyclic permutation of all the single bit states, the division by l takes care of the over counting. In the evaluation of the gauge theory correlator, one has to sum over the distinct cyclic permutations in the trace. Thus the combinatorics involved in the evaluation of the 2-string overlap is identical to the that of planar Wick contractions, the functional dependence is governed by the product of the corresponding single bit overlaps. Therefore we conclude that the 2-string overlap in (3.33) reproduces the corresponding two point function of the gauge invariant operator.

To illustrate the combinatorics involved we consider two simple examples.

Example (i). Consider the most simple case with all the $\hat{O}^{(s)}$ being equal to the operator $\exp(ixP)U$, that is every one of the single bit state is the vacuum state at position x. The gauge theory operator corresponding to this state is given by

$$\sum_{\pi}\prod_{i=1}^{l}\left(O(x)^{i}|0\rangle^{(i)}\right) \leftrightarrow \frac{1}{\sqrt{N^{l}}}\mathrm{Tr}(Z^{l}) \tag{3.37}$$

The 2-string overlap of this state is given by

$$\frac{1}{l}\sum_{\pi\sigma}\prod_{i=1}^{l}{}^{(i)}\langle 0|U^{\dagger}\exp(-iPy)\exp(iPx)U|0\rangle^{(i)} = \frac{l^{2}}{l}\frac{1}{(x-y)^{2l}}. \tag{3.38}$$

Note that there is no sign involved in the permutation as all operators are bosonic. Since all operators in each single bit Hilbert space is identical the sum over π and σ permutations just give a factor of l^2, which cancels with the denominator to give the right power of l. (3.38) is identical to the corresponding two point function of the operator in (3.37) A point to emphasize is that, as in (3.37), we normalize all gauge theory operators so that the planar two point functions do not have any N dependence.

Example (ii). Consider the string state corresponding to the operator $\frac{1}{\sqrt{N^l}}\mathrm{Tr}(\phi^i z^{l-1})$ given in (2.37) and (2.38). Evaluating the 2-string overlap of this state we obtain

$$\begin{aligned}\frac{1}{l}\sum_{\pi\sigma}\prod_{i=1}^{l} {}^{(i)}\langle 0|O^{\pi_i\dagger}(x)O^{\sigma_i}(y)|0\rangle^{(i)} &= \frac{1}{l}\sum_{\pi}\prod_{i=1}^{l} {}^{(i)}\langle 0|O^{\sigma_i\dagger}(x)O^{\sigma_i}(y)|0\rangle^{(i)}, \\ &= \frac{1}{(x-y)^{2l}}.\end{aligned} \tag{3.39}$$

In the first line we have used the fact that the only when both permutation σ and π are the same the overlap is non-zero. Then the summation of the permutations gives a factor of l which cancels with factor of l in the denominator. Finally, we have used the single bit overlap for the scalars. Thus the last line is the expected two point function for the operator under consideration.

3.3 Three-point functions

Here we construct the 3-string overlap which reproduces the planar 3-point functions of the free theory. Our discussion is divided into two parts, 3-point functions for which length of the operators is conserved and 3-point functions for which the length is not conserved. The 3-vertex constructed for the length conserving processes has a natural interpretation of two strings joining into a third string such that the length of the third string is the sum of the lengths of the first and the second. This is similar to the 3-vertex of light cone string field theory of the critical string. The length non-conserving process the 3-vertex can also be interpreted as string joining process, except the the first and the second string also overlap with each other.

Length conserving process. We first consider three point functions for which the length of one of the operator say $\hat{\mathcal{O}}_3$ is the sum of the lengths of the remaining two. We define the overlap of these states by the following,

$$\frac{1}{N}\langle 0|\hat{\mathcal{O}}_1^\dagger(x_1)\otimes\langle 0|\hat{\mathcal{O}}_2^\dagger(x_2)\hat{\mathcal{O}}_3(x_3)|0\rangle, \tag{3.40}$$

where $\langle 0|\hat{\mathcal{O}}_1^\dagger(x_1)$ is a state in the Hilbert space $\mathcal{H}^{(l_1)}$, $\langle 0|\hat{\mathcal{O}}_2^\dagger(x_2)$ a state in the Hilbert space $\mathcal{H}^{(l_2)}$ and $\hat{\mathcal{O}}^3(x_3)|0\rangle$ a state in the Hilbert space $\mathcal{H}^{(l_3)}$ and $l_1+l_2=l_3$. $\otimes$ just refers to the tensor product of the states. The inner product in (3.40) is the usual bit by bit inner product. We have indicated this diagrammatically in figure (2). This overlap rule will be more familiar when written in terms of a state in the three string Hilbert space. It is given by

$$|V_3\rangle = \frac{1}{N}\exp\left[\sum_{s=1}^{l^{(1)}}(a^{\dagger(1)}(s)a^{\dagger(3)}(s)+b^{\dagger(1)}(s)b^{\dagger(3)}(s)\right.$$

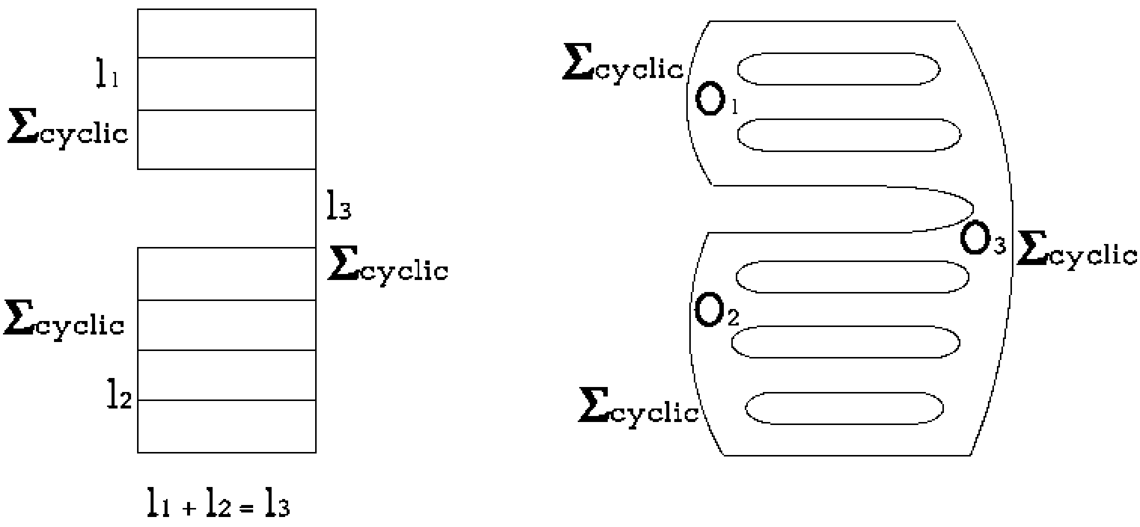

Figure 2: The three vertex and the 3-point function, length conserving process.

$$
\begin{aligned}
&+\alpha^{\dagger(1)}(s)\alpha^{\dagger(3)}(s)+\beta^{\dagger(1)\dagger}(s)\beta^{\dagger(3)}(s)) \qquad (3.41)\\
&+\sum_{s=l^{(1)}+1}^{l^{(3)}}(a^{\dagger(2)}(s)a^{\dagger(3)}(s)+b^{\dagger(2)}(s)b^{\dagger(3)}(s)\\
&+\alpha_s^{\dagger(2)}\alpha_s^{\dagger(3)}+\beta_s^{\dagger(2)}\beta_s^{\dagger(3)})\Big]\,|0\rangle_{(1)}\otimes|0\rangle_{(2)}\otimes\rangle_{(3)}.
\end{aligned}
$$

Here the superscripts over the oscillators just indicate the Hilbert space they belong to and s indicates the bit label. It is clear from the discussion of the 2-vertex, this vertex basically implements the delta function overlap $\delta(X^{(1)}+X^{(2)}-X^{(3)})$ where the X's indicate the world sheet coordinate of the bit string. In the above vertex, the bits from 1 to $l^{(1)}$ of the 1st string overlaps the corresponding bits of the third string and the $l^{(2)}$ bits of the second string labeled from $l^{(1)}+1$ to $l^{(3)}$ overlaps the corresponding bits on the third string. Note that as expected we have normalized the three vertex by $1/N$ as it is the closed string coupling. It is also easy to see from the discussion of the previous subsection we have

$$
\frac{1}{N}\langle 0|\hat{\mathcal{O}}_1^\dagger(x_1)\otimes\langle 0|\hat{\mathcal{O}}_2^\dagger(x_2)\hat{\mathcal{O}}_3(x_3)^\dagger|O\rangle=\langle 0|\hat{\mathcal{O}}_1^\dagger(x_1)\otimes\langle 0|\hat{\mathcal{O}}_2^\dagger(x_3)\langle 0|\hat{\mathcal{O}}_3^\dagger(x_3)|V_3\rangle. \qquad (3.42)
$$

From the figure (2), comparison of this overlap rules to Wick contractions involved in obtaining the corresponding free and planar Yang-Mills three point function indicate that both the functional dependence of the correlation function as well as the structure constants will be reproduced by the 3-string overlap. The overlaps rules just mimic planar Wick contractions. The conformal Ward identities satisfied by the three-point functions are guaranteed to be satisfied, due to the fact that the single bit overlap satisfies the Ward identities, this will ensure the right functional dependence. In the appendix D we evaluate the 3-string overlap for a class of scalars and show that both the functional dependence as well as the structure constants is in agreement with the gauge theory results.

JHEP04(2006)014

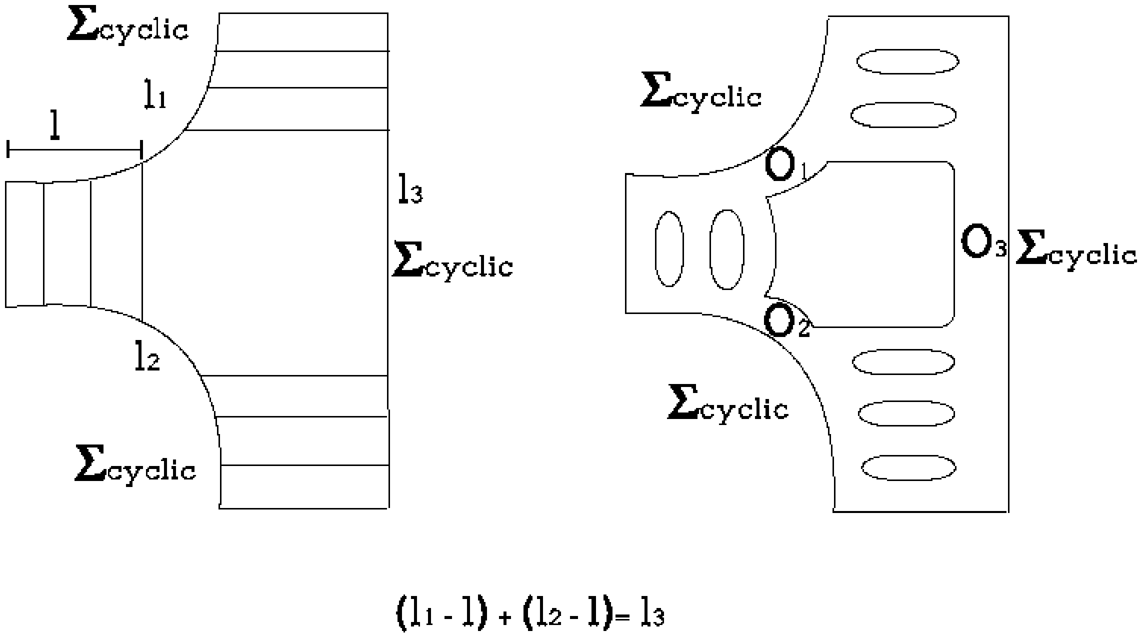

Figure 3: The three vertex and the 3-point function, length non-conserving process.

Length non-conserving process. Consider three operators $\hat{\mathcal{O}}_i(x_i)$ with lengths $l^{(3)} \geq l^{(2)} \geq l^{(1)}$. It is easy to see that for free Wick contractions such operators will have nontrivial three point functions only if $2l = l^{(1)} + l^{(2)} - l^{(3)}$ and l is an integer. The overlap rules for the states corresponding to these operators are most clearly seen in the first diagram of figure (3). We overlap l bits of the states $\hat{\mathcal{O}}(x_1)|0\rangle$ and $\hat{\mathcal{O}}(x_2)|0\rangle$ using the usual bit inner product. Then overlap the remaining $l^{(1)} - l$ bits of $\hat{\mathcal{O}}(x_1)|0\rangle$, and $l^{(2)} - l$ bits of $\hat{\mathcal{O}}(x_2)$ with that of the operator $\hat{\mathcal{O}}(x_3)$ as shown in the figure. It is also clear from figure (3) that this rule mimics planar Wick contractions of the corresponding operators in the gauge theory. Furthermore, the sum over the cyclic permutations of the bits corresponds to the sum over all the distinct planar Wick contractions possible due to the cyclic property of the trace. Again since the single bit overlap satisfies the conformal Ward identities the 3-string overlap constructed out of single bit overlaps will satisfy the respective conformal Ward identities. It is possible to write the overlap rule for the length non-conserving process as the following state in the three Hilbert space

$$
\begin{aligned}
|V_3\rangle &= \exp\left[N^{(13)} + N^{(23)} + N^{(12)}\right] |0\rangle_1 \otimes |0\rangle_2 \otimes |0\rangle_3 \frac{1}{N}, \qquad (3.43)\\
N^{(13)} &= \sum_{s=1}^{l^{(1)}-l} a^{(1)\dagger}(s) a^{(3)\dagger}(s) + b^{(1)\dagger}(s) b^{(3)\dagger}(s) + \alpha^{(1)\dagger}(s)\alpha^{(3)\dagger}(s) + \beta^{(1)\dagger}(s)\beta^{(3)\dagger}(s),\\
N^{(23)} &= \sum_{s=l^{(1)}-l+1}^{l^{(3)}} \Big(a^{(2)\dagger}(s+l) a^{(3)\dagger}(s) + b^{(2)\dagger}(s+l) b^{(3)\dagger}(s)\\
&\qquad + \alpha^{(2)\dagger}(s+l)\alpha^{(3)\dagger}(s) + \beta^{(2)\dagger}(s+l)\beta^{(3)\dagger}(s)\Big),
\end{aligned}
$$

$$N^{(12)} = \sum_{s=l^{(1)}-l+1}^{l^{(1)}} \Big(a^{(1)\dagger}(2l^{(1)}+1-s-l)a^{(2)\dagger}(s) + b^{(1)\dagger}(2l^{(1)}+1-s-l)b^{(2)\dagger}(s)$$
$$+ \alpha^{(1)\dagger}(2l^{(1)}+1-s-l)\alpha^{(2)\dagger}(s) + \beta^{(1)\dagger}(2l^{(1)}+1-s-l)\beta^{(2)\dagger}(s) \Big) .$$

This 3-vertex implements the overlap rule whereby l bits of the states $\hat{\mathcal{O}}_1(x_1)|0\rangle$ and $\hat{\mathcal{O}}(x_2)|0\rangle$ overlaps with each other and the remaining $l^{(1)} - l$ and $l^{(2)} - l$ bits of these states overlaps with the state $O_3(x_3)|0\rangle$. Note that in the 3-vertex (3.43), the bits of the second string are labeled from $l^{(1)} - l + 1$ to $l^{(3)} + l$, thus there are $l^{(2)} = l^{(3)} - l^{(1)} + 2l$ bits. As in the discussion of the length conserving process, the three point function of the states $\hat{\mathcal{O}}_1(x_1)|0\rangle, \hat{\mathcal{O}}_2(x_2)|0\rangle, \hat{\mathcal{O}}_3(x_3)|0\rangle$ is given by

$$\langle 0|\hat{\mathcal{O}}\ \dagger_1\ (x_1) \otimes \langle 0|\hat{\mathcal{O}}_2^\dagger(x_2)\langle 0|\hat{\mathcal{O}}_3^\dagger(x_3)|V_3\rangle. \tag{3.44}$$

In the appendix D we evaluate the three-point function for a length non-conserving process and show that both the functional dependence as well as the structure constants is in agreement with the gauge theory results.

4. String overlap at one-loop in λ

From the previous section we see that the two-point functions and the three-point functions at $\lambda = 0$ were essentially determined by the following: i) The position dependence of the correlators was governed by the equation $U^\dagger U E = -EU^\dagger U$. This equation guaranteed conformal Ward identities which are required to be satisfied for the correlation functions. ii) The combinatorics involved in the planar Wick contractions was reproduced in the bit picture by the delta function overlap rule. This guaranteed that the structure constants in the bit picture was identical to that of the gauge theory. In this section, we derive the modifications to the above picture when on rendering λ finite. From the AdS/CFT dictionary (1.1), we see that rendering α' finite would introduce interactions between bits. At first order in λ and in the planar limit, only the nearest neighbour bits would interact. Therefore, turning on λ modifies the free propagation of bits in the bit string theory. The nearest neighbour interactions should be such that it reproduces the logarithmic corrections as well as the finite corrections to structure constants of the three-point functions. We now outline the strategy by which we will introduce these nearest neighbour interactions between the bits so as to reproduce the logarithmic divergences and the structure constants at one loop in λ.

Logarithmic corrections. From the discussion in section 3.1 of the single bit overlap and, in particular, from the method of conformal Ward identities, we see that the position dependence essentially arose from to the identity $U^\dagger U E = -EU^\dagger U$. Examining the quantities involved in the identity in some detail will provide the clue of how to reproduce the logarithms. E is the operator conjugate to the dilatation operator (2.15). At $\lambda = 0$, for a bit string of length l we see that E is the sum of the conformal dimensions of all the letters composing the string, therefore from the discrete world sheet point of view it is a global charge on the world sheet. As we have argued from (1.1), we see that turning on λ renders α' finite, which introduces interaction between the bits at one loop. At the planar level, this interaction just involves only nearest neighbours at one-loop in λ. Therefore, the global charge E gets corrected to $E + \lambda\sum_{s=1}^{l} H_{s,s+1}$, where $H_{s,s+1}$ is the anomalous dimension Hamiltonian which contains the information of the interactions. Similarly, $U^\dagger U = \exp(\frac{\pi}{2}(P_0 - K_0))$, where P_0 and K_0 are global charges on the discrete world sheet. Again at one loop in λ the charges $P_0 - K_0$ gets corrected to $P_0 - K_0 + \lambda\sum_{s=1}^{l}\delta K_{0s,s+1}$. Here we have chosen a scheme in which the global Lorentz generators are not corrected and the interactions are present only in the corrections of the special conformal generators. We will show that it is possible to choose such a scheme. The corrections H and δK_μ are such that they have to respect global $SO(2,4)$ invariance to one-loop in λ. This will ensure that the identity $U^\dagger U E = -E U^\dagger U$ is satisfied to one-loop in λ, which in turn guarantees conformal Ward identities at one-loop and thus reproduces the logarithmic divergences. We will explicity implement this strategy in this section to obtain the logarithmic corrections in both the two-point function and the three-point functions in the bit string picture.

Structure constants. To indicate how to incorporate the corrections to structure constants, let us recall how they arise in the gauge theory. Consider a set of conformal primary operators O_i.[9] By conformal invariance, the general form for the two-point functions of these operators at one-loop in λ and at large N is given by

$$\langle O_i(x_1)O_j(x_2)\rangle = \frac{1}{(x_1-x_2)^{2\Delta_i}}\left(\delta_{ij} + \lambda g_{ij} - \lambda\gamma_i\delta_{ij}\ln((x_1-x_2)^2\Lambda^2)\right). \tag{4.1}$$

Note that at one-loop one also obtains a finite constant mixing matrix g_{ij} proportional to λ. Though this matrix in scheme dependent, it contributes through a scheme independent combination to the one-loop corrections to the structure constants. We now detail the precise combination by which it contributes to the one loop corrections to the structure constants. The three point function of these operators at one loop is given by

$$\begin{aligned}\langle O_i(x_1)O_j(x_2)O_k(x_3)\rangle = {} & \frac{1}{|x_{12}|^{\Delta_i+\Delta_j-\Delta_k}|x_{12}|^{\Delta_i+\Delta_k-\Delta_j}|x_{23}|^{\Delta_j+\Delta_k-\Delta_i}}\times \\ & \left(C^{(0)}_{ijk}\left[1-\lambda\gamma_i\ln|\frac{x_{12}x_{13}\Lambda}{x_{23}}| - \lambda\gamma_j\ln|\frac{x_{12}x_{23}\Lambda}{x_{13}}|\right.\right. \\ & \left.\left. -\lambda\gamma_i\ln|\frac{x_{13}x_{23}\Lambda}{x_{12}}|\right] + \lambda\tilde{C}^{(1)}_{ijk}\right).\end{aligned} \tag{4.2}$$

Here again the finite constant $\tilde{C}^{(1)}_{ijk}$ is not renormalization scheme independent, but the following combination is the scheme independent correction to the structure constant

$$C^{(1)}_{ijk} = \tilde{C}^{(1)}_{ijk} - \frac{1}{2}g_{ii'}C^{(0)}_{ijk} - \frac{1}{2}g_{jj'}C^{(0)}_{ij'k} - \frac{1}{2}g_{kk'}C^{(0)}_{ijk'}. \tag{4.3}$$

A more detailed discussion of the one loop corrections to structure constants is given in [30].

[9] We will work with scalars and choose a basis such that their anomalous dimension matrix is diagonal for simplicity.

We will now argue the that the metric corrections g_{ij} can be understood as the changes in the inner product of the states built with oscillators a, b, α, β . A convenient way to think of the metric correction g_{ij} in the two-point function is that it is a correction to the scalar product of the corresponding states. It can be isolated from the logarithmic corrections by evaluating the two-point function with the operator O_j at $x_2 = 0$ and the operator $f \circ O_i(0)$, where f refers to the inversion $x^\mu \to -x^\mu/x^2$. Evaluating the two-point function we then obtain

$$\begin{aligned} \langle f \circ O_i(0) O_j(0) \rangle &= \lim_{x \to 0} \frac{1}{x^{2\Delta_i + \lambda\gamma_i}} x^{2\Delta_i + \lambda\gamma_i} \left(\delta_{ij} + \lambda g_{ij}\right), \\ &= \delta_{ij} + \lambda g_{ij}. \end{aligned} \tag{4.4}$$

Note that here we have performed the inversion also at one-loop in λ to cancel the logarithmic corrections. We will refer to this two-point functions as to the norm. In the bit language performing the inversion on one of the states is equivalent to acting on it with the operator $U^2 = \exp(\pi M_{05})$, which corresponds to the inversion (3.20). Then the norm of the operators O_i and O_j is given by

$$\begin{aligned} \langle 0| O_i^\dagger(0) U^2 O_j(0) |0\rangle &= \langle 0| s_i^\dagger U^\dagger U^2 U s_j |0\rangle, \\ &= \langle 0| s_i^\dagger s_j |0\rangle, \end{aligned} \tag{4.5}$$

where $O_j(0)$ refers to the state in the string bit Hilbert space. To obtain the first line of the above equation we use the definition that a state in the string bit Hilbert space is a obtained by the action of U on the fock space of oscillators. Here we have denoted this state by s_i, s_j. For instance the s_i's for the basic letters of $\mathcal{N} = 4$ YM is given in the first column of table 3. To obtain the last line we have used the fact that $U^\dagger = U$ and $U^4 = 1$. Thus the norm of two operators is just the inner product in the Fock space of the oscillators. This fact is more obvious when one considers only the scalar $SO(6)$ sector, since the in this sector the states are created only by the fermionic α, β oscillators and the norm is entirely governed by the inner product of these fermionic oscillators.

We need a strategy to incorporate the change in the norm at one loop in λ. At first sight it would be hard to imagine one can consistently deform the inner product of the oscillators to incorporate the nearest neighbour interactions. This deformation also should respect the global $PSU(2,2|4)$ symmetry of the theory. In critical string theory compactified on a circle in the direction 9 say, the norm of the oscillators in the direction of the circle is given by $\langle \alpha_1^9 \alpha_{-1}^9 \rangle = G^{99}$. Here the change in the norm of the oscillator α_9 is due the change in the corresponding kinetic term. This provides us the clue to incorporate the change in the norm. We must first write down a world-sheet Lagrangian for the oscillators a, b, α, β with nearest neighbour interactions proportional to λ such that the quantization of the Lagrangian gives the usual inner product along with a correction proportional to λ. This will provide us a consistent deformation of the norm. A natural choice for the world sheet

Hamiltonian is the operator E [10], the world sheet Lagrangian for the $\lambda = 0$ case is given by

$$\mathcal{S} = \sum_s^l \int dt \frac{i}{2}\left(\bar{\psi}_s \partial_t \psi_s - \partial_t \bar{\psi}_s \psi_s\right) + \frac{1}{2}\psi_s^\dagger \psi_s + \frac{i}{2}\left(\varphi_s^\dagger \partial_t \varphi_s - \partial_t \varphi_s^\dagger \varphi_s\right). \tag{4.6}$$

Here ψ and φ refer to the $SO(2,4)$ spinor and $SO(6)$ spinor defined in (2.9) and (2.21) respectively. Note that the potential term $\psi^\dagger\psi$ breaks global $SO(2,4)$ invariance [11], it is clear that this is necessary as the Hamiltonian is one of the generators of $SO(2,4)$. Performing canonical quantization on this action we obtain the world sheet Hamiltonian

$$\mathcal{H} = \frac{1}{2}\sum_s^l \psi_s^\dagger \psi_s = \frac{1}{2}\sum_s (a_s^\dagger a_s + b_s b_s^\dagger) = E \tag{4.7}$$

The commutation relations of the oscillators are the ones given in (2.10) and (2.22). As we have discussed earlier for the case of critical string theory compactified on the circle, a world-sheet deformation which modifies the commutation relations of the oscillators $\{a, b, \alpha, \beta\}$ must be proportional to the kinetic energy term in the Lagrangian (4.6). In section 4.2, for the $SU(2)$ subsector, we will introduce such a deformation in the Lagrangian which modifies the commutation relations of the oscillators to include interactions with nearest neighbour oscillators. We show that such a deformation is unique upto a proportionality constant and it preserves the global $SU(2)$ symmetry. We then use the approach developed by [38] to construct the deformed two-string vertex and the three-string vertex. Using these new vertices we show that the structure constants obtained from the bit picture in the $SU(2)$ sector agree with those of the gauge theory evaluated in [30], up to a proportionality constant.

4.1 Logarithmic corrections at one loop

Two point functions. As we discussed earlier, at first order in λ we have nearest neighbour interactions between the bits. We have indicated this correction to the string bit overlap for the two-point functions schematically in figure (4). From this figure it is clear that it suffices to focus on the corrections to the two bit overlap, just as for the case of $\lambda = 0$ it was sufficient to focus on the single bit overlap. The first step in our strategy is to evaluate the corrections to the global charges P_μ and K_μ. It is most convenient to write the corrected charges in terms of differential operators acting on the oscillators states. For simplicity we will work in the $SO(6)$ subsector, though the discussion can be easily generalized to all sectors. The charges at one-loop in λ for a two bit state in the $SO(6)$ sector is given by

$$\begin{aligned} P_\mu \exp(-ixP))|s_1 s_2\rangle &= -i\partial_\mu \exp(-ixP)|s_1 s_2\rangle, \\ K_\mu \exp(-ixP)|s_1 s_2\rangle &= i(2(\Delta^{(0)} + \lambda H_{12})x_\mu \end{aligned} \tag{4.8}$$

[10] One can possibly assume that there exists a a gauge in which the world sheet Hamiltonian is the conformal dimension of operators just as in the case of the plane wave limit where $\Delta - J$ is the light cone Hamiltonian.

[11] $\bar{\psi}\psi$ is $SO(2,4)$ invariant

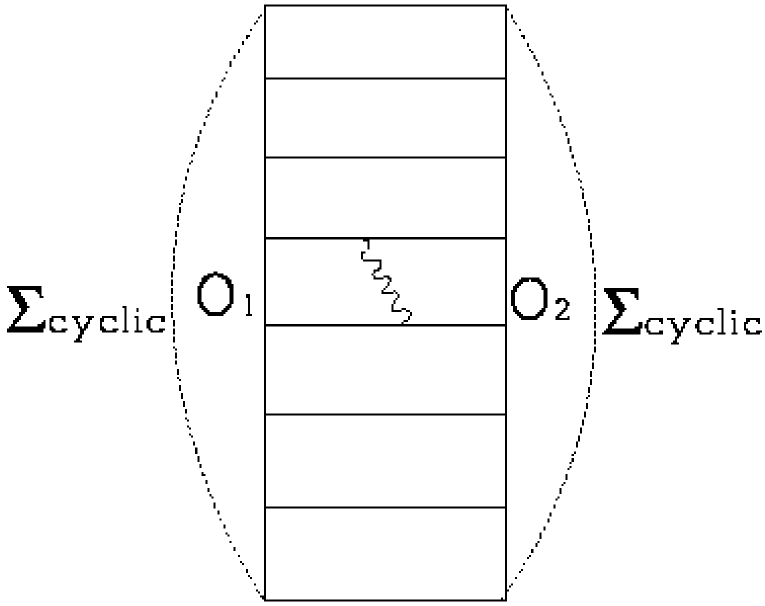

Figure 4: Nearest neighbour interactions in two-string overlap

$$
\begin{aligned}
&+2x_\mu x\cdot\partial - x^2\partial_\mu)\exp(-ixP)|s_1s_2\rangle,\\
&= (K_\mu^{(0)} + 2\lambda H_{12}x_\mu)\exp(-ixP)|s_1s_2\rangle,\\
E\exp(-ixP)|s_1s_2\rangle &= ((\Delta^{(0)} + \lambda H_{12}) + x\cdot\partial)\exp(-ixP)|s_1s_2\rangle,\\
&= (E^{(0)} + \lambda H_{12})\exp(-ixP)|s_1s_2\rangle.
\end{aligned}
$$

Here $\Delta^{(0)} = 2$ for the two bit state $|s_1s_2\rangle$, and s_1s_2 refer to the two $SO(6)$ labels of the states. This state can be written in terms of the fermionic oscillators α, β acting on the two bit vacuum. $P = P_1 + P_2$ is the global momentum of the two bits and similarly K stands for the global special conformal generator of the two bits. H_{12} refers to the anomalous dimension Hamiltonian which has the information of the interactions between the two bits, it can be written in terms of the fermionic oscillators. It is easily shown that these corrected generators satisfy the conformal algebra (2.1) to first order in λ. In doing this one uses the fact that H_{12} is a Casimir of the λ=0 algebra [39], therefore in the manipulations we can treat H_{12} as a c-number. Let us now show that the identity $U^\dagger UE = -EU^\dagger U$ is also true to one loop when U is constructed out of the corrected generators given in (4.8). We first evaluate $U^\dagger U$ to first order in λ

$$
\begin{aligned}
U^\dagger U &= \exp\left(\frac{\pi}{2}(P_0 - K_0^{(0)} - 2i\lambda x_0 H_{12})\right), \qquad (4.9)\\
&= \exp\frac{\pi}{2}\left(P_0 - K_0^{(0)}\right)\\
&\quad -i\lambda H_{12}\int_0^\pi dt\exp(\frac{t}{2}(P_0^x - K_0^{(0)}))x_0\exp\left(\frac{1-t}{2}(P_0^x - K_0^{(0)})\right).
\end{aligned}
$$

Here we have used the following expansion, which is valid to first order in λ, for any two operators A and B.

$$\exp(A+\lambda B) = \exp(A) + \int_0^1 dt \exp(tA)\lambda B \exp((1-t)A) + O(\lambda^2). \tag{4.10}$$

In (4.9) we have also used the fact that H_{12} is the Casimir of the algebra at $\lambda = 0$ to move it to the extreme left. Now, to evaluate the integral on the last line in (4.9) we use the following relation

$$\begin{aligned}
&\int_0^\pi dt \exp(\frac{t}{2}(P_0 - K_0^{(0)}))x_0 \exp(-\frac{t}{2}(P_0 - K_0^{(0)})) \\
&= \int_0^\pi dt \frac{2x_0\cos t + i(1-x^2)\sin t}{2ix_0\sin t + (1-x^2)\cos t + (1+x^2)}, \\
&= -i\log(x^2).
\end{aligned} \tag{4.11}$$

The simplest way to obtain the first line of the above equation is to realize that $P_0 - K_0 = 2M_{05}$ and use the geometric action of the rotation M_{05} on x_0. This action is not linear, but M_{05} acts linearly on the light cone coordinates introduced in [37] as a rotation. Using the relation of the light cone coordinates to x^μ one obtains the relation in (4.11). Note that the occurrence of i in the above equation is due to the fact that transformation $\exp(tM_{05})$ acts as a boost but with imaginary angle. Furthermore, for convenience, we have worked in units so that the coordinates are dimensionless. Reinstating the dimensions of the coordinates would give a scale in the logarithm of the above equation. Substituting (4.11) in (4.9) we obtain the formula for $U^\dagger U$ to linear order in λ

$$\begin{aligned}
U^\dagger U &= (1 - \lambda H_{12}\log(x^2))\exp\frac{\pi}{2}(P_0 - K_0^{(0)}), \\
&= (1 - \lambda H_{12}\log(x^2))(U^\dagger U)^{(0)}.
\end{aligned} \tag{4.12}$$

It is now possible to verify the relation $EU^\dagger U = -U^\dagger UE$ to first order in λ. The steps involved are indicated below

$$\begin{aligned}
EU^\dagger U &= (\Delta^{(0)} + x\cdot\partial + \lambda H_{12})U^\dagger U, \\
&= (\Delta^{(0)} + x\cdot\partial + \lambda H_{12})(1 - \lambda H_{12}\log(x^2))(U^\dagger U)^{(0)}, \\
&= U^\dagger U(-\Delta^{(0)} - x\cdot\partial - 2\lambda H_{12} + \lambda H_{12}), \\
&= -U^\dagger UE.
\end{aligned} \tag{4.13}$$

The ingredients we have used in the manipulations are the fact that the zeroth order operator $E^{(0)}$ anti-commutes with $(U^\dagger U)^{(0)}$, H_{12} is a Casimir and finally $[x\cdot\partial, \log(x^2)] = 2$. In fact it is this last fact which gives the extra $-2\lambda H_{12}$ in the third line of the above equation. We have of course retained only terms to order λ.

It is clear from the method of conformal Ward identities of the previous section, that, once the $U^\dagger UE = -EU^\dagger U$ is true to order λ, the equation (3.10) becomes

$$(x\partial_x + y\partial_y + 2\Delta^{(0)} + 2\lambda H_{12})I(x,y) = 0. \tag{4.14}$$

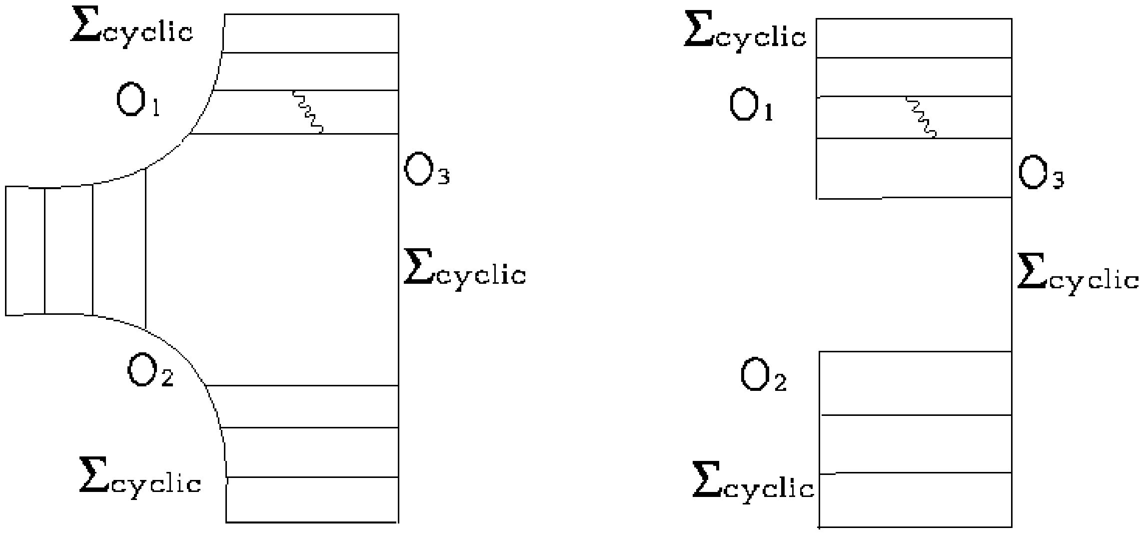

Figure 5: Two body corrections to 3-string overlap

This together with translation invariance results in the following solution for the two-point function for the two bit states $|s_1 s_2\rangle$ at x and $|s_1' s_2'\rangle$ at y

$$I(x,y) = \frac{1}{(x-y)^{2\Delta^{(0)}}}(1 - 2\lambda\Delta\log(x-y)^2). \tag{4.15}$$

In the above equation Δ refers to the c-number $\langle s_1 s_2|H_{12}|s_1' s_2'\rangle$. This is the required two-bit overlap of the two-bit states, which on summation over all the nearest neighbour bits reproduces the required logarithmic corrections in the two-point functions. Thus the overlap with the corrected U as given in (4.12) reproduces the logarithmic corrections in the two-point functions.

Three-point functions. It is clear that once the logarithmic divergences of all primaries in the two-point functions are fixed, then from conformal invariance one can also determine all the logarithmic divergences in the three-point functions. Here we want to see how these arise in the three-string overlap. As before, to keep the discussions simple, we will restrict our attention to the $SO(6)$ subsector, but it can be easily generalized to all primaries.

The figures (5) and (6), schematically indicate the nearest neighbour interactions that need to be included in the three string overlaps for both the length conserving process and the length non-conserving process. The interactions can be divided into two basic types: (i) Two body interactions, which involve interactions between nearest neighbour bits belonging to any two of the strings participating in the three string overlap. These are shown in figure (5). (ii) Genuine three body interactions as shown in figure (6)

The logarithmic corrections for the two-body interactions are identical to the corrections encountered for the two-point functions. The reason for this is as follows: let the two bit-strings involved in the two-body interactions be at positions x_1 and x_2. Then the nearest neighbour bits in each of these strings are at these respective positions. The corrections

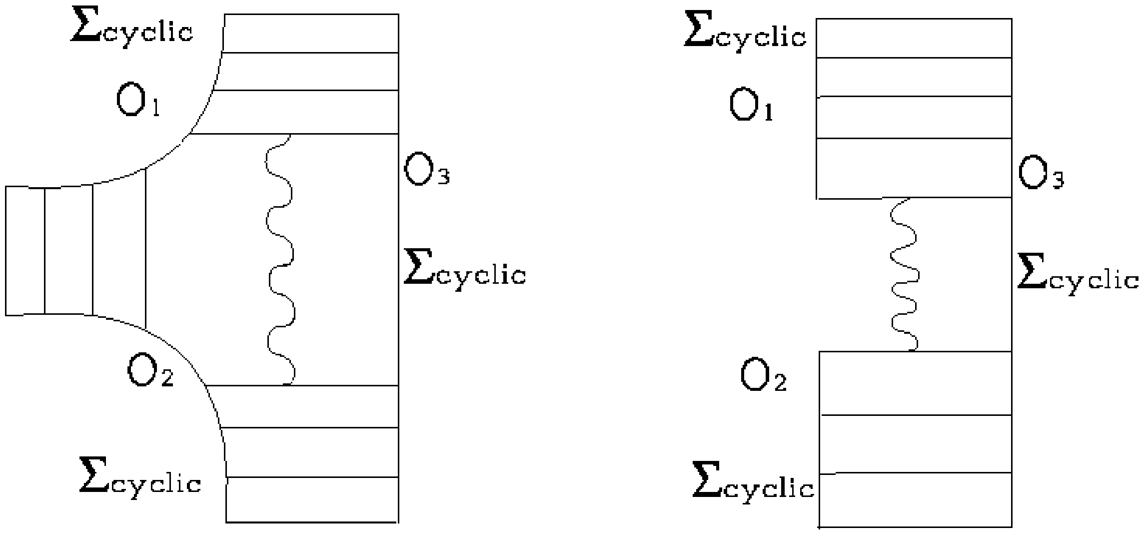

Figure 6: Genuine three body corrections to 3-string overlap

to $U^\dagger U$ are then identical to that encountered for the two-point function. The discussion of the previous section goes through and the logarithmic corrections are identical to those for the two-bit case, discussed in the previous section.

As for the genuine three-body interactions, it is clear that one cannot just use the corrections to $U^\dagger U$ to evaluate the logarithmic divergences to the overlap. This is because two of the bits belong to two different strings and therefore they are at two different positions. Then the corrections to the charges cannot be just written as those given in (4.8), which assumes that the two bits are the same position. The generic three-body interaction will involve two bits at two different positions, which belong to two different strings, say string O_1 and O_2, and two nearest neighbour bits which belong to the third string O_3, as indicated in the figure (6). The strategy we will employ to obtain the corrections to the overlap is, first, to obtain the two bits at different positions as an expansion of the same bits at a common position and then perform the overlap onto the two nearest neighbour bits of the third string O_3. When $\lambda = 0$, this expansion is just a simple Taylor series expansion. But, at finite λ, when the bits are at the same position, they carry anomalous dimensions. The simple Taylor series expansion does not respect this scaling and therefore we have find the appropriate expansion, which is the usual operator product expansion, given, for example, in [37]. In the language of the bits it is given by

$$\begin{aligned} & U_1 \exp\left(-ix_1P_1\right)|s_1\rangle U_2 \exp\left(-ix_2P_2\right)|s_2\rangle = \\ & x_{12}^{\lambda H_{12}} F(1+\frac{1}{2}\lambda H_{12}; 2+\lambda H_{12}; -(x_{12})\cdot\partial_2) U \exp(-ix_2(P_1+P_2)|s_1\rangle|s_2\rangle + \cdots \end{aligned} \tag{4.16}$$

The subscripts on the operators U and P just refer to the site at which the operator acts. $F(a;c;z)$ refers to the confluent hypergerometric function, which has the following integral representation

$$F(a;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 du\cdot u^{a-1}(1-u)^{c-a-1}e^{uz} \tag{4.17}$$

Note that on the left hand side of the equation (4.16) the bits are at the same position x_2 and the operator U acts on both the sites. In performing the operation $(x_{12}\cdot\partial_2)^n$ only the symmetric and traceless components are retained. The dots in (4.16) refer to the tensor primaries at higher levels, and we have also kept only the contribution from the scalar and its conformal descendents. This is sufficient if one is interested in only three point functions of operators belonging to the $SO(6)$ subsector. This is because, finally we have to take the inner product of the expansion of the left-hand side of (4.16) with two bits consisting of only scalars at position x_3. Therefore only the scalar component in the expansion survives. Since the operator on the left-hand side of (4.16) is at the same position, we can now evaluate the inner product with two nearest neighbour bits of the third string, obtaining

$$\begin{aligned} T(s_1', s_2', s_1, s_2) &= \langle s_1'|\langle s_2'|\exp(ix_3P^\dagger)U^\dagger U_1\exp(-ix_1P_1)|s_1\rangle U_2\exp(-ix_2P_2)|s_2\rangle, \\ &= \langle s_1'|\langle s_2'|x_{12}^{\lambda H_{12}}\textstyle\int_0^1 duu^{\lambda\frac{H_{12}}{2}}(1-u)^{\lambda\frac{H_{12}}{2}}\exp(-ux_{12}\cdot\partial_{x_2})\frac{1}{x_{32}^{4+2\lambda H_{12}}}|s_1\rangle|s_2\rangle. \end{aligned} \tag{4.18}$$

Note that the overlap is identical to that occurring in the two-point function. To obtain the second line, we have substituted the equation (4.15). Performing the u integral we obtain

$$T(s_1', s_2', s_1, s_2) = \frac{1}{x_{13}^2x_{23}^2}(1-\lambda\Delta(\ln|x_{13}|+\ln|x_{23}|-\ln|x_{12}|). \tag{4.19}$$

where $\Delta = \langle s_1'|\langle s_2'|H_{12}|s_1\rangle|s_2\rangle$. The integral in (4.18) is evaluated by first expanding all the terms one order in λ. The easiest way to argue the result of the integral is that given in (4.19), is to choose a holomorphic direction and set $x_{12} = z_{12}$. Then the exponential with the derivative just acts as a translation with the parameter $-uz_{12}$. This is because choosing a holomorphic direction automatically ensures that there is no trace in the action of the derivative. Now it is easy to act this on the function $1/x_{34}^{4+2\lambda\Delta}$ and perform the integrations. Then one can reinstate the value of z_{12} as x_{12} to obtain the above result. The result in (4.19) is the required answer between for the genuine three body interactions to reproduce the logarithms in the three-point functions. Thus we have shown that the overlap at one loop in λ reproduces the logarithms for the three-point functions.

4.2 Structure constants in the $SU(2)$ subsector

As argued in the introduction to this section, to obtain the structure constants we need to deform the commutation relations of the oscillators $\{a, b, \alpha, \beta\}$. The consistent way to do this is to obtain the deformed relations from a world-sheet action. In this subsection we show that in the $SU(2)$ subsector one can uniquely determine the nearest neighbour interactions in the world sheet that reproduce the structure constants to one loop.

The $SU(2)$ subsector consists of gauge invariant operators made of two complex scalars belonging to two different Cartans of $SO(6)$, say z and y. In terms of the oscillators, only one of the α's and one of the β's are excited in this subsector. The generators for the $SU(2)$ subsector are given by

$$J^i = \bar{\varphi}\sigma^i\varphi \tag{4.20}$$

where σ^i refers to the Pauli matrices and φ is the $SU(2)$ spinor given by

$$\varphi = \begin{pmatrix} \alpha \\ -\beta^\dagger \end{pmatrix}, \qquad \bar{\varphi} = \left(\alpha^\dagger, -\beta\right). \tag{4.21}$$

The operators corresponding to the two states are $z \to |0\rangle$ and $y \to \alpha^\dagger\beta^\dagger|0\rangle$. Note that the combination $\bar{\varphi}\varphi$ is $U(1)_Z$ invariant.

To obtain the corrections to the structure constants we start with the following bit world sheet Lagrangian in the $SU(2)$ sector.

$$\begin{aligned}\mathcal{L} = \sum_s^l \Big[&\frac{i}{2}\left(\bar{\varphi}_s\partial_t\varphi_s - \partial_t\bar{\varphi}_s\varphi_s\right) \\ &+ \lambda_1\frac{i}{2}(\bar{\varphi}_s\partial_t\sigma^i\varphi_s - \partial_t\bar{\varphi}_s\sigma^i\varphi_s)\left(\bar{\varphi}_{s+1}\sigma^i\varphi_{s+1} + \bar{\varphi}_{s-1}\sigma^i\varphi_{s-1}\right) \\ &+ \lambda_2\frac{i}{2}(\bar{\varphi}_s\partial_t\varphi_s - \partial_t\bar{\varphi}_s\varphi_s)\left(\bar{\varphi}_{s+1}\varphi_{s+1} + \bar{\varphi}_{s-1}\varphi_{s-1}\right) \\ &+ \lambda V(\varphi_s\varphi_{s+1})\Big].\end{aligned} \tag{4.22}$$

As argued in the introduction to this section, we need to modify the kinetic energy term in the action with nearest neighbour interactions so that we will obtain deformed commutation relations for the oscillators, which will in turn change their norm. The terms proportional to λ_1 and λ_2 in (4.22) are the most general nearest neighbour interactions consistent with the following symmetries:

- Local $U(1)_Z$ symmetry. All terms occur in combinations $\bar{\varphi}_s\varphi_s$ which respects local $U(1)_Z$ symmetry.
- Orientation preserving symmetry. That is, the interaction with the $s+1$ th site is the same as the interaction with the $s-1$ th site.
- Global $SU(2)$ symmetry. In (4.22) we have parameterized the interaction into two components. λ_1 refers to the strength of the spin 1 interaction between the nearest neighbours, and λ_2 refers to the strength of the spin 0 interaction between the nearest neighbours. Both λ_1 and λ_2 will in general be proportional to the 't Hooft coupling.

In (4.22), the potential $V(\varphi_s, \varphi_{s+1})$, refers to the anomalous dimension Hamiltonian. This is because we require the world sheet Hamiltonian to be the operator E, which measures the conformal dimension. The zero point energy of the a, b oscillators already provide the bare dimensions of the operators in the $SO(6)$ subsector. For the first order action given in (4.22), the Hamiltonian is just given by the potential $V(\varphi_s, \varphi_{s+1})$. Therefore, the potential is the anomalous dimension Hamiltonian. In the next section we will provide a gauging principle which determines this potential, here we just focus on the structure constants.

Our strategy will be first to quantize the Lagrangian given in (4.22), and determine the modified commutation relations. Then we construct the vertex which implements the delta function overlap consistent with the modified commutation relations, and finally evaluate the overlap of the states. We will show that the structure constants are identical, upto to an overall constant, to those obtained in the gauge theory calculation of [30]

The canonical conjugate momenta to φ_x and $\bar{\varphi}_x$ are given by

$$\begin{aligned}
\Pi_{\varphi_s^a} &- \left[\frac{i}{2}\bar{\varphi}_s^a + \frac{i\lambda_1}{2}(\bar{\varphi}_s\sigma^i)_a\left(\bar{\varphi}_{s+1}\sigma^i\varphi_{s+1} + \bar{\varphi}_{s-1}\sigma^i\varphi_{s-1}\right)\right. \\
&\quad \left. + \frac{i\lambda_2}{2}\bar{\varphi}_s^a\left(\bar{\varphi}_{s+1}\varphi_{s+1} + \bar{\varphi}_{s-1}\varphi_{s-1}\right)\right] = 0, \\
\Pi_{\bar{\varphi}_s^a} &- \left[\frac{i}{2}\varphi_s^a + \frac{i\lambda_1}{2}(\sigma^i\varphi_s)_a\left(\bar{\varphi}_{s+1}\sigma^i\varphi_{s+1} + \bar{\varphi}_{s-1}\sigma^i\varphi_{s-1}\right)\right. \\
&\quad \left. + \frac{i\lambda_2}{2}\varphi_s^a\left(\bar{\varphi}_{s+1}\varphi_{s+1} + \bar{\varphi}_{s-1}\varphi_{s-1}\right)\right] = 0,
\end{aligned} \tag{4.23}$$

where $a, b, \ldots$ takes values $1, 2$. We will denote the first and the second equations as $C_{\varphi_s^a} = 0$ and $C_{\bar{\varphi}_s^a} = 0$ respectively. These are second class constraints as their Poisson brackets are non-zero. To quantize a Lagrangian with second class constraints, we need to use the Dirac brackets. To this end, let us compute the commutation relations among the constraints. They are given by

$$\begin{aligned}
\{C_{\varphi_s^a}, C_{\varphi_t^b}\} = 0, \qquad &\{C_{\bar{\varphi}_s^a}, C_{\bar{\varphi}_t^b}\} = 0, \\
\{C_{\varphi_s^a}, C_{\bar{\varphi}_t^b}\} &= \delta_{st}\delta_{ab} \\
&+ \lambda_1\delta_{st}\sigma^i_{ba}\left(\bar{\varphi}_{s+1}\sigma^i\varphi_{s+1} + \bar{\varphi}_{s-1}\sigma^i\varphi_{s-1}\right) \\
&+ \lambda_1(\sigma^i\varphi_t)_b\left(\delta_{t+1,s}(\bar{\varphi}_{t+1}\sigma^i)_a + \delta_{t-1,s}(\bar{\varphi}_{t-1}\sigma^i)_a\right) \\
&+ \lambda_2\delta_{st}\delta_{ba}\left(\bar{\varphi}_{s+1}\varphi_{s+1} + \bar{\varphi}_{s-1}\varphi_{s-1}\right) \\
&+ \lambda_2\varphi_t^b\left(\delta_{t+1,s}\bar{\varphi}^a_{t+1} + \delta_{t-1,s}\bar{\varphi}^a_{t-1}\right).
\end{aligned} \tag{4.24}$$

The Dirac bracket is then given by [40]

$$\{\bar{\varphi}_s^a, \varphi_t^b\}_{\mathrm{DB}} = \{\bar{\varphi}_s^a, \varphi_t^b\} - \{\bar{\varphi}_s^a, C_{\bar{\varphi}_u^c}\}\{C_{\bar{\varphi}_u^c}, C_{\varphi^d_{u'}}\}^{-1}\{\varphi^d_{u'}, \phi_t^b\}. \tag{4.25}$$

Here summation over u, u', c, d is implied, and we have also used the fact that the matrix of the commutation relations of the constraints is off diagonal (4.24). Substituting the last equation of (4.24) and evaluating the inverse in (4.25) upto first order in λ's, we obtain

$$\begin{aligned}
\{\bar{\varphi}_s^a, \varphi_t^b\}_{\mathrm{DB}} &= \delta_{s,t}\delta_{a,b} \\
&- \lambda_1\delta_{st}\sigma^i_{ba}\left(\bar{\varphi}_{s+1}\sigma^i\varphi_{s+1} + \bar{\varphi}_{s-1}\sigma^i\varphi_{t-1}\right) \\
&- \lambda_1(\sigma^i\varphi_t)_b\left(\delta_{t+1,s}(\bar{\varphi}_{t+1}\sigma^i)_a + \delta_{t-1,s}(\bar{\varphi}_{t-1}\sigma^i)_a\right) \\
&- \lambda_2\delta_{st}\delta_{ba}\left(\bar{\varphi}_{s+1}\varphi_{s+1} + \bar{\varphi}_{s-1}\varphi_{s-1}\right) - \lambda_2\varphi_t^b\left(\delta_{t+1,s}\bar{\varphi}^a_{t+1} + \delta_{t-1,s}\bar{\varphi}^a_{t-1}\right).
\end{aligned} \tag{4.26}$$

The rest of the commutators vanish

$$\{\varphi_s^a, \varphi_t^b\}_{\mathrm{DB}} = 0, \qquad \{\bar{\varphi}_s^a, \bar{\varphi}_t^b\}_{\mathrm{DB}} = 0. \tag{4.27}$$

From now on, (anti-)commutation relations involving the φ's will always be understood as Dirac brackets. To prove that these relations are consistent, one can show that the

JHEP04(2006)014

following Jacobi identities are satisfied to linear order in λ's.

$$[\varphi_u^c, \{\bar{\varphi}_s^a, \varphi_t^b\}] + [\bar{\varphi}_s^a, \{\varphi_t^b, \varphi_u^c\}] + [\varphi_t^b, \{\varphi_u^c, \bar{\varphi}_s^a\}] = 0 \tag{4.28}$$
$$[\bar{\varphi}_u^c, \{\bar{\varphi}_s^a, \varphi_t^b\}] + [\bar{\varphi}_s^a, \{\varphi_t^b, \bar{\varphi}_u^c\}] + [\varphi_t^b, \{\bar{\varphi}_u^c, \bar{\varphi}_s^a\}] = 0.$$

To construct the delta function overlaps, it is convenient to work with oscillators which are diagonal either in the position space or in the momentum space. To this end we define the oscillators χ which satisfy the usual commutation rules.

$$\begin{aligned}
\chi_s^a &= \varphi_s^a + \frac{\lambda_1}{2}(\sigma^i\varphi_s)_a\left(\bar{\varphi}_{s+1}\sigma^i\varphi_{s+1} + \bar{\varphi}_{s-1}\sigma^i\varphi_{s-1}\right) \\
&+ \frac{\lambda_2}{2}\varphi_s^a\left(\bar{\varphi}_{s+1}\varphi_{s+1} + \bar{\varphi}_{s-1}\varphi_{s-1}\right), \\
\bar{\chi}_s^a &= \bar{\varphi}_s^a + \frac{\lambda_1}{2}((\bar{\varphi}_s\sigma^i)_a\left(\bar{\varphi}_{s+1}\sigma^i\varphi_{s+1} + \bar{\varphi}_{s-1}\sigma^i\varphi_{s-1}\right) \\
&+ \frac{\lambda_2}{2}\bar{\varphi}_s^a\left(\bar{\varphi}_{s+1}\varphi_{s+1} + \bar{\varphi}_{s-1}\varphi_{s-1}\right).
\end{aligned} \tag{4.29}$$

With these definitions, it is easy to verify that the commutation relations of the χ's to first order in λ are given by the usual relations

$$\{\bar{\chi}_s^a, \chi_t^b\} = \delta_{st}\delta_{ab}, \quad \{\chi_s^a, \chi_t^b\} = 0, \quad \{\bar{\chi}_s^a, \bar{\chi}_t^b\} = 0. \tag{4.30}$$

From the definition of the χ's in (4.29) it is clear that the vacuum state for the φ oscillators remains a vacuum state for the χ oscillators. It is also useful to have the inverse relations

$$\begin{aligned}
\varphi_s^a &= \chi_s^a - \frac{\lambda_1}{2}(\sigma^i\chi_s)_a\left(\bar{\chi}_{s+1}\sigma^i\chi_{s+1} + \bar{\chi}_{s-1}\sigma^i\chi_{s-1}\right) \\
&- \frac{\lambda_2}{2}\chi_s^a\left(\bar{\chi}_{s+1}\chi_{s+1} + \bar{\chi}_{s-1}\chi_{s-1}\right), \\
\bar{\varphi}_s^a &= \bar{\chi}_s^a - \frac{\lambda_1}{2}((\bar{\chi}_s\sigma^i)_a\left(\bar{\chi}_{s+1}\sigma^i\chi_{s+1} + \bar{\chi}_{s-1}\sigma^i\chi_{s-1}\right) \\
&- \frac{\lambda_2}{2}\bar{\chi}_s^a\left(\bar{\chi}_{s+1}\chi_{s+1} + \bar{\chi}_{s-1}\chi_{s-1}\right).
\end{aligned} \tag{4.31}$$

Note that, using the χ oscillators, it is easy to construct the global $SU(2)$ generators to the leading order in the λ's. They are given by $\sum_s^i \bar{\chi}(s)\sigma^i\chi(s)$.

To obtain the corrected structure constants, following [38], we define the overlap using the diagonal oscillators χ's. To be more explicit it is convenient to work with the $SU(2)$ invariant vacuum given by

$$\begin{aligned}
|0\rangle' &= -\beta^\dagger|0\rangle, \qquad \text{then} \quad \varphi^a|0\rangle' = 0, \\
\bar{\varphi}^a|0\rangle' &= \alpha^\dagger\beta^\dagger|0\rangle, \qquad \text{for} \;\; a=1, \\
&= |0\rangle \qquad\qquad \text{for} \;\; a=2,
\end{aligned} \tag{4.32}$$

It is clear, using (4.31), that the vacuum $|0\rangle'$ remains the vacuum for the χ oscillators. This is because the λ corrections in (4.31) involve annihilation operators χ, which can be replaced by φ to the leading order in λ. It is sufficient for us to deal with the two bit overlap for the sites s and $s+1$. This is given by

$$_{s+1}\langle 0|'_s\langle 0|'\chi_{s+1}^d\chi_s^c\bar{\chi}_s^a\bar{\chi}_{s+1}^b|0\rangle'_s|0\rangle'_{s+1} = \delta^{ca}\delta^{db}. \tag{4.33}$$

The sites s and $s+1$ can be on bits belonging to two different strings. Such a situation will occur in a three-body term. To obtain the λ dependence of the two-point functions or three-point functions in string theory, we need to evaluate the overlap constructed out of diagonal oscillators χ, as in (4.33), on the old states constructed out of the φ oscillators [38]. We now give the results for the overlap of two-bit states.

$$\begin{aligned}\langle 0|\varphi^d_{s+1}\varphi^c_s\bar{\varphi}^a_s\bar{\varphi}^b_{s+1}|0\rangle &= \delta^{ca}\delta^{db} \\ &\quad - \lambda_1(\sigma^i_{ca})_s(\sigma^i_{db})_{s+1} - \lambda_2\delta_{ca}\delta_{db},\end{aligned} \tag{4.34}$$

To evaluate the overlap, we write the φ oscillators in terms of the χ oscillators using (4.31) and then apply the definition of the overlap given in (4.33). We can now determine the relationship between the λ's using the fact that on chiral primaries there should be no corrections. This implies that, when all the indices in (4.34) are same the λ, the corrections have to vanish. This condition gives $\lambda_2 = -\lambda_1 = \lambda$. We thus obtain

$$\langle 0|\varphi^d_{s+1}\varphi^c_s\bar{\varphi}^a_s\bar{\varphi}^b_{s+1}|0\rangle = \delta^{ca}\delta^{db} + \lambda(1-\sigma^i_s\sigma^i_{s+1})_{ca,db}. \tag{4.35}$$

Note that the corrections to the norm are proportional to the anomalous dimension Hamiltonian of the $SU(2)$ sector which is given by $\sum_s(1-\sigma_s\cdot\sigma_{s+1})$. To compute structure constants one needs to evaluate the renormalization group invariant quantity given in (4.3). As shown in [30] the corrections to the structure constants from all the two body terms in the three point functions will vanish on subtracting the metric corrections. What remains are half the contribution of the genuine three body terms. The three body corrections can be read out from (4.35). Therefore we find that the structure constants in the $SU(2)$ sector is dictated by the anomalous dimension Hamiltonian, which was the result in the gauge theory calculation of [30]. Here we have arrived at this conclusion entirely from the symmetries in the bit picture and using the formalism of [38] to obtain the corrections to the overlap.

5. Anomalous dimensions for the $SU(2)$ sector.

Throughout this paper we have emphasized the role of the oscillator variables for a string bit formulation of $\mathcal{N}=4$ Yang-Mills. One of the underlying symmetries in this formulation is the local $U(1)_Z = N_a+N_\alpha-N_b-N_\beta$. In section 2.3 we saw that the spectrum was obtained after gauging this symmetry. As we discussed in the previous section, this symmetry also played a role in determining the structure constants in the $SU(2)$ subsector, at one loop in λ. In this section we present a linear $U(1)_Z$ gauged model for the oscillators φ^a, which reproduces the anomalous dimension Hamiltonian to two loops. To be precise, we define the anomalous dimension Hamiltonian to be given by

$$\exp c\left(\sum_{n=1}^{\infty}\lambda^n H^{(n)}\right) = \int \mathcal{D}U\exp(\sum_s^l g(\varphi^\dagger_s U_{s,s+1}\varphi_{s+1}+\text{h.c})). \tag{5.1}$$

where $U_{s,s+1}$ are $U(1)$ link variables. Under local $U(1)_Z$ transformation the link variables transform as $U_{s,s+1}\to\exp(i\theta_s)U_{s,s+1}\exp(-i\theta_{s+1})$, thus the Hamiltonian on the left-hand side has local $U(1)_Z$ symmetry. It has global $SU(2)$ symmetry and is also just a linear model in the spinor variables φ_s. Only nearest neighbour oscillators are coupled in the model. The integration over the link variables is just the classical integration and $\mathcal{D}U = \prod_s^l \frac{1}{2\pi}(U_s^{-1}dU_s)$. The coupling g depends on the 't Hooft parameter λ. It will turn out to be proportional to the square root of the 't Hooft coupling. c is an overall constant. In fact, the above model is a simplified version of the one considered in [41, 42] to obtain the Heisenberg spin chain.

At first sight it is not immediately obvious that one would obtain next nearest neighbour interaction of the kind required for the two-loop anomalous dimension Hamiltonian, since the model involves only a nearest neighbour coupling. Also, it is not obvious one would obtain a local Hamiltonian after integrating the link variables. In this section we integrate out the link variables in (5.1) to two loops and show that we obtain the two-loop anomalous dimension Hamiltonian for the $SU(2)$ subsector, as found in [43] from a gauge theory calculation. We also show that the gauged linear model obeys BMN scaling to all loops. In deriving both these facts we neglect the contributions of the modified commutation relations obtained in the previous section. These would in principle contribute from two loop and above. A complete analysis would require to keep track of their contribution at all loops. We leave this for later work, but independently of this, one could ask the question if the model in (5.1) reproduces the three-loop anomalous dimension Hamiltonian found in [44, 45] [12]. Unfortunately the answer turns out to be negative, but in the appendix E we show that the inclusion of non-nearest couplings in (5.1) is sufficient to reproduce the anomalous dimension Hamiltonian to three loops. It would be interesting to see if the simple model presented in (5.1), together with the modified commutation rules due to just the nearest neighbour interactions introduced in (4.22), would reproduce the anomalous dimension to three loops.

5.1 Anomalous dimensions to two loops

Consider the linear model in (5.1) to two loops

$$\exp(c(\lambda H^{(1)} + \lambda H^{(2)})) = \int \mathcal{D}U \exp(g \sum_s^l (A_s U_{s,s+1} + A_s^\dagger U_{s,s+1})) \tag{5.2}$$

where $A_s = \bar{\varphi}_s \varphi_{s+1} = \alpha_s^\dagger \alpha_{s+1} + \beta_s \beta_{s+1}^\dagger$ and c is an overall constant. As an illustration of the calculations involved we first obtain the one-loop answer in detail. We start by expanding the exponential and then perform the U integrations. Due to the U integrations only even powers of g are retained, and for the one loop answer it is sufficient to retain only the second order term. This is given by

$$\begin{aligned} \tilde{H}^{(1)} &= \int \mathcal{D}U \frac{g^2}{2} \sum_{s,t} \left(A_s A_t^\dagger + A_s^\dagger A_t \right) U_{s,s+1} U_{t,t+1}^\dagger, \\ &= \frac{g^2}{2} \sum_s (A_s A_s^\dagger + A_s^\dagger A_s). \end{aligned} \tag{5.3}$$

[12] We thank Jan Plefka who emphasized on testing the model to three loops.

On external states, which satisfy the $U(1)_Z$ constraint, it is easy to see that $A_s^\dagger A_s = A_s A_s^\dagger$. This is due to the commutation relation

$$[A_s, A_t^\dagger] = \delta_{st} G_s = \delta_{st}\left(N_s^\alpha - N_s^\beta - (N_{s+1}^\alpha - N_{s+1}^\beta)\right), \tag{5.4}$$

where N_s^α, N_s^β refer to the number operators of the oscillators α, β respectively. Note that the right hand side of this equation is the $U(1)_Z$ charge, which vanishes on physical external states. We will refer to this as the Gauss' law constraint G. Therefore the last line of equation (5.3) reduces to

$$\begin{aligned}\tilde{H}^{(1)} &= \frac{g^2}{2}\sum_s A_s A_s^\dagger, \\ &= \frac{g^2}{2}\sum_s (1 - \sigma_s \cdot \sigma_{s+1}).\end{aligned} \tag{5.5}$$

In the last line of the above expression we have expressed the operator $A_s A_s^\dagger$ in terms of Pauli matrices. To do this one uses the identifications of J^+, J^-, J^3 in terms of oscillators given in (4.20) and the $U(1)_Z$ constraint. Equating the leading order expansion on the left hand side of (5.2), we find

$$c\lambda H^{(1)} = \frac{g^2}{2}\sum_s (1 - \sigma_s \sigma_{s+1}). \tag{5.6}$$

We have thus obtained, up to a proportionality constant, the one loop anomalous dimension Hamiltonian found in [46] for the $SU(2)$ subsector. See [12] for a recent review on developments in this subject.

We now proceed to expand (5.2) to order g^4 to obtain the two-loop Hamiltonian. At this order one encounters the following

$$\begin{aligned}\tilde{H}^{(2)} &= \frac{g^4}{4!}\int \mathcal{D}U \left[\sum (A_s U_{s+1} + A_s^\dagger U_{s+1}^\dagger)\right]^4 \\ &= \frac{g^4}{4!}\left[\sum_{s,t}\left(A_s A_t A_s^\dagger A_t^\dagger + A_s A_t A_t^\dagger A_s^\dagger\right) - \sum_s A_s A_s A_s^\dagger A_s^\dagger \right. \\ &\quad \left. +(5 \ \text{Permutations})\right]\end{aligned} \tag{5.7}$$

In the second line we have performed the U integrations, which retains only terms with equal number of A's and $A^\dagger$'s. The permutations refer to all other possible arrangements of 2 A's and 2 $A^\dagger$'s. Using the commutation relation in (5.4) we can bring all the other permutations to the standard form, in which all the A's are to the left and $A^\dagger$'s are to the right. In performing this operations there are additional terms which arise due to right hand side of (5.4). It is easy to see that all these terms cancel. We will illustrate this with the following term which is the permutation of the standard term

$$\sum_{s,t}\left(A_s A_s^\dagger A_t A_t^\dagger + A_s A_t^\dagger A_t A_s^\dagger\right) - \sum_s A_s A_s^\dagger A_s A_s^\dagger \tag{5.8}$$

$$= \sum_{s,t}\left(A_sA_tA_s^\dagger A_t^\dagger + A_sA_tA_t^\dagger A_s^\dagger\right) - \sum_s A_sA_sA_s^\dagger A_s^\dagger$$
$$- \sum_s A_sG_sA_s^\dagger - \sum_s A_s\sum_t G_tA_s^\dagger + \sum_s A_sG_sA_s^\dagger.$$

Here G_s refers to the Gauss' law constraint defined in (5.4).The last line contains the terms obtained after moving the A's to the left. Note that first and the third terms of the last line cancel each other and the middle term vanishes identically, since $\sum_y G_y = 0$. Using this procedure, all the five permutations in (5.7) can be brought to the standard form.

What remains now is to write the standard form in terms of Pauli matrices. This is done by bringing together the A's and $A^\dagger$ which have the same position index. Once they are together we can easily write the various terms as Pauli matrices. Performing these manipulations we have

$$\tilde{H}^{(2)} = \frac{6g^4}{4!}\left(2\sum_s A_sA_s^\dagger\sum_t A_tA_t^\dagger + 2\sum_s A_sG_sA_s^\dagger + \frac{1}{2}\sum_{s,t}[A_s,A_t][A_t^\dagger A_s^\dagger]\right). \tag{5.9}$$

We have not included the term $A_sA_sA_s^\dagger A_s^\dagger$, as it can be shown to vanish on any external state satisfying the $U(1)_Z$ constraint. To evaluate the extra terms we need the following commutation relations

$$\begin{aligned}[][G_s,A_t^\dagger] &= (\delta_{s,t+1}+\delta_{t,s+1}-2\delta_{s,t})A_t^\dagger, \\ [A_s,A_t] &= \delta_{s+1,t}A_s^{(2)} - \delta_{s,t+1}A_t^{(2)}, \\ [A_s^\dagger,A_t^\dagger] &= \delta_{t+1,s}A_t^{(2)\dagger} - \delta_{t,s+1}A_s^{(2)\dagger}.\end{aligned} \tag{5.10}$$

where $A_s^{(2)} = \bar{\varphi}_s\varphi_{s+2} = \alpha_s^\dagger\alpha_{s+2} + \beta_s\beta_{s+2}^\dagger$. This involves oscillators of next to nearest neighbour positions. Substituting the above relations in (5.9) and using the fact that when the Gauss' law G is on the extreme right it vanishes on physical external states, we obtain

$$\begin{aligned}\tilde{H}^{(2)} &= \frac{g^4}{2}\sum_{s,t}A_sA_s^\dagger A_tA_t^\dagger + \frac{g^4}{4}\sum_s\left(-4A_sA_s^\dagger + A_s^{(2)}A_s^{(2)\dagger}\right), \\ &= \frac{g^4}{2}\left[\sum_s\frac{1}{2}(1-\sigma_s\cdot\sigma_{s+1})\right]^2 + \frac{g^4}{4}\sum_s\left(-4\frac{1}{2}(1-\sigma_s\cdot\sigma_{s+1}) + \frac{1}{2}(1-\sigma_s\cdot\sigma_{s+2})\right), \\ &= \frac{1}{2}c^2\lambda^2(H^{(1)})^2 + c\lambda^2H^{(2)}.\end{aligned} \tag{5.11}$$

Note that the non-local term is precisely the one due to the expansion of the one-loop term in the exponential. The local term is proportional to the two-loop anomalous dimension Hamiltonian. The relative coefficient of the nearest neighbour term and the next nearest neighbour term ensures BMN scaling. To obtain the two-loop anomalous dimension Hamiltonian, one then chooses $c\lambda = g^2/2$ and $c\lambda^2 = g^4/8$ which gives $\lambda = g^2/4$ and $c = 2$. [13]

[13] Using a similar gauged linear model for the $SU(1|1)$ subsector we obtain the leading $1/J$ piece of the 2-loop anomalous dimension Hamiltonian found in [47 – 51]. Restricting the model to the $SU(2|3)$ subsector we have verified that we obtain the 1-loop anomalous dimension Hamiltonian for this subsector.

BMN scaling to all loops. We now show that the nearest neighbour interaction in (5.1) gives rise to anomalous dimensions satisfying BMN scaling to all loops. [14] Recall that the nearest neighbour coupling $\bar{\varphi}_s U_{s,s+1}\varphi_{s+1}+\text{h.c.}$ in (5.1) can be written as $A+A^\dagger$ where $A=\sum_s A_s U_{s,s+1}$ and $A^\dagger=\sum_s A_s^\dagger U_{s,s+1}^\dagger$, we first note, using (5.4) $[A,A^\dagger]=0$. This allows one to write $e^{\sqrt{\lambda}V^{(1)}}=e^{\sqrt{\lambda}A^\dagger}.e^{\sqrt{\lambda}A}$. The n-loop term then comes from the operator

$$\frac{1}{n!^2}\lambda^n\int \mathcal{D}U\, A^{\dagger^n}A^n. \tag{5.12}$$

Our strategy will be to determine the representation of this operator on physical Hilbert space. In the BMN limit the relevant states are the ones with large J that have small number of impurities distributed sparsely. Let us denote the two $SU(2)$ doublet states as $|0\rangle$, which we shall call ground state and $\alpha^\dagger\beta^\dagger|0\rangle$ which will be called the impurity at a given site. In the BMN limit the distance between any two impurities is greater than n where n is the order of perturbation we are interested in. Since we will finally be taking the local terms in the interaction (the non-local terms canceling from the lower order terms as explained above), it is sufficient to focus on just one impurity say at site s. This state denoted by $|\psi_s\rangle$ is then $\alpha_s^\dagger\beta_s^\dagger|0\rangle$ where $|0\rangle=\prod_t|0\rangle_t$. It is easy to see that the action of A on $|\psi_s\rangle$ is

$$A|\psi_x\rangle=\alpha_{s-1}^\dagger U_{s-1,s}\beta_s^\dagger|0\rangle-\alpha_s^\dagger U_{s,s+1}\beta_{s+1}^\dagger|0\rangle\equiv|\psi_{s-1,s}\rangle-|\psi_{s,s+1}\rangle, \tag{5.13}$$

where

$$|\psi_{s,s+k}\rangle=\alpha_s^\dagger\prod_{j=0}^{k-1}U_{s+j,s+j+1}\beta_{s+k}^\dagger|0\rangle. \tag{5.14}$$

By repeatedly applying A we obtain

$$A^n|\psi_s\rangle=\sum_{k=0}^{n}(-1)^{n-k}\ {}^nC_k|\psi_{x-k,x+n-k}\rangle. \tag{5.15}$$

Using the fact that the inner products

$$\langle\psi_{s,s+k}|\psi_{t,t+m}\rangle=\delta_{st}\delta_{km}, \tag{5.16}$$

we obtain for $0\le m\le 2n$,

$$\begin{aligned}\langle\psi_{s-n+m}|A^{\dagger^n}A^n|\psi_s\rangle &= (-1)^{n+m}\sum_{k=|n-m|}^{n}{}^nC_k\ {}^nC_{k-|n-m|},\\ &=\frac{1}{2\pi i}\int\frac{dz}{z}z^{|n-m|}(1-z)^n(1-\frac{1}{z})^n,\\ &=\frac{(-1)^n}{2\pi i}\int\frac{dz}{z}z^{|n-m|-n}(1-z)^{2n}=(-1)^{m-n}\ {}^{2n}C_m,\end{aligned} \tag{5.17}$$

[14] We thank Jan Plefka for showing that the linear model in (5.1) satisfies BMN scaling to 5-loops numerically.

where the z integral is along a contour surrounding the origin. For m outside this range the inner product above is zero. Using the completeness relation we then find

$$A^{\dagger n} A^n |\psi_s\rangle = \sum_{m=0}^{2n} (-1)^{m-n} \; {}^{2n}C_m |\psi_{x-n+m}\rangle. \tag{5.18}$$

To see that this result satisfies BMN scaling we go to the states in the momentum basis

$$|p\rangle = \sum_s e^{2\pi i p s/J} |\psi_s\rangle, \tag{5.19}$$

where J is the total length of the spin chain. Then (5.18) implies

$$\begin{aligned} \int \mathcal{D}U e^{\sqrt{\lambda} V_1} |p\rangle, &= \sum_n \frac{\lambda^n}{(n!)^2} \int \mathcal{D}U A^{\dagger n} A^n |p\rangle \\ &= \sum_n \frac{\lambda^n}{(n!)^2} \sum_{m=0}^{2n} (-1)^{m-n} \; {}^{2n}C_m e^{2\pi i p(m-n)/J} |p\rangle, \\ &= \sum_n \frac{\lambda^n}{(n!)^2} (2i \sin \frac{\pi p}{J})^{2n} |p\rangle \to \sum_n \frac{1}{(n!)^2} (\frac{-4\pi^2 \lambda p^2}{J^2})^n |p\rangle, \end{aligned} \tag{5.20}$$

where the last relation is in the large J limit. Note that because of the coefficients $(-1)^{m-n} \; {}^{2n}C_m$ in (5.18), all the terms involving less powers of $1/J^2$ cancel and the leading term goes like $1/J^{2n}$. This is the correct n-loop BMN scaling behaviour that goes like $(\lambda/J^2)^n$. The effective Hamiltonian which is the logarithm of the above expression clearly also obeys the BMN scaling. We can then ask whether the leading terms that survive, give rise to the BMN anomalous dimension formula. The answer unfortunately turns out to be negative for three and more loops. In any case as, we will show in appendix E, the model in (5.1) with only the nearest interaction does not give the right kind of terms that appear in the gauge theory 3-loop anomalous dimension. We need to include also terms that are bilinear in φ and connect $\bar{\varphi}$ and φ at next nearest neighbour and next-next nearest sites. However as we mentioned in the introduction to this section, we have not taken into account the modified commutation rules given in (4.26). This would in principle contribute from 2-loop onwards. It will be interesting to see if the model with just nearest neighbour interactions given in (5.1) together with the modified commutation rules due to the nearest neighbour interactions given in (4.22) is sufficient to reproduce the 3-loop anomalous dimensions.

6. Conclusions

In this paper we have shown that the $\mathcal{N} = 4$ Yang-Mills theory at weak coupling can be formulated as a theory of a discretized string, whose world-sheet σ direction is a one-dimensional lattice of points, the string bits. The degrees of freedom of the world-sheet consists of 8 bosonic and 8 fermionic oscillators at each lattice site. The spectrum of gauge invariant words of the Yang-Mills theory at $\lambda = 0$ is identical to the spectrum of states in the Hilbert space of these oscillators, together with certain $U(1)$ gauge and cyclicity constraints. We showed how to obtain the two-point and three-points functions of all gauge invariant words as the simple delta function overlap of the discrete world-sheet. At first order in λ, i.e. at one loop in the gauge theory, we showed that logarithmic corrections and structure constants can be incorporated as nearest neighbour interactions between the string bits. In fact, for the $SU(2)$ subsector, we could determine the structure constants from the symmetries of the bit picture, upto to an overall proportionality constant. The symmetries of the bit picture also enabled us to obtain the two-loop anomalous dimension Hamiltonian from a $U(1)$ gauged linear model of the corresponding oscillators. This model had only nearest neighbour interactions and it obeyed BMN scaling to all loops.

There are several open problems: at a more technical level, it would be of some interest to see how the modified commutation relations introduced in section 4 for the $SU(2)$ subsector alter the calculations of section 5 for the two-loop anomalous dimensions Hamiltonian.

More generally, throughout this paper we always worked in the planar limit of the gauge theory, which corresponds to the tree-level of the discretized string theory. An interesting question would be to study the $1/N^2$ corrections to the (free) two-point and three-point functions, to see if they arise as a genus one contribution in the discrete world-sheet theory. In [16 − 18], the correlations functions in the free theory was rewritten to appear as string like amplitudes in AdS. It would be interesting to see if there is any relationship between this and the string bit formulation of this paper.

Perhaps the most pertinent and obvious question that arises is whether the discretized string bit model discussed in this paper admits a continuum limit.

Acknowledgments

J.R.D. would like to thank Avinash Dhar and Rajesh Gopakumar for useful discussions. L.F.A. thanks Gleb Arutyunov for reading the manuscript and suggesting improvements. We thank Jan Plefka for useful correspondences, suggestions and interesting discussions. We would like to thank the organizers for the Third Regional Crete Meeting in String Theory for the opportunity to present this work. The work of the authors is partially supported by the RTN European program: MRTN-CT-2004-503369.

A. Notations and conventions

We first summarize the conventions for the indices

$A, B, \ldots$	$0,1,2,3,5,6$: $SO(2,4)$ indices.
$\mu, \nu, \ldots$	$0,1,2,3$: 4d space time indices.
$m, n, \ldots$	$1,2,3$ space indices.
$I, J, \ldots$	$1,2,3,4,5,6$: $SO(6)$ R-symmetry indices.
$i, j, \ldots$	$1,2,3,4$: $SO(4) \subset SO(6)$ indices.

JHEP04(2006)014

$\gamma, \delta, \ldots$	$SU(2)_R \subset SO(1,3)$ spinorial indices.
$\dot{\gamma}, \dot{\delta}, \ldots$	$SU(2)_L \subset SO(1,3)$ spinorial indices.
$\tau, \upsilon, \ldots$	$SU(2)_{L'} \subset SO(4) \subset SO(6)$ R-symmetry spinorial indices.
$\dot{\tau}, \dot{\upsilon}, \ldots$	$SU(2)_{R'} \subset SO(4) \subset SO(6)$ R-symmetry spinorial indices.
$s, t, \ldots$	Site labels on bit string.

We work in the signature diag$(-,+,+,+,+,-)$ for $SO(2,4)$. Raising and lowering the $\gamma, \dot{\gamma} \ldots$ indices will be done as in [36].

Four dimensional representation of $SO(2,4)$ generators. Our conventions for the four dimensional Weyl representation of $SO(2,4)$ generators is as follows.

$$\begin{aligned} M^{\mu\nu} &= -\frac{i}{4}[\gamma^\mu, \gamma^\nu], \qquad & M^{\mu 5} &= -\frac{i}{2}\gamma^\mu\gamma^5, \\ M^{\mu 6} &= \frac{1}{2}\gamma^\mu, \qquad & M^{56} &= \frac{1}{2}\gamma^5. \end{aligned} \tag{A.1}$$

Here the γ matrices are 4×4, $SO(1,3)$ gamma matrices in the the Dirac representation they obey the algebra $\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}$, where $\eta^{\mu\nu} = \text{diag}(-1,1,1,1)$. These matrices are given by

$$\begin{aligned} \gamma^0 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ \gamma^m &= \begin{pmatrix} 0 & \sigma^m \\ -\sigma^m & 0 \end{pmatrix}, \\ \gamma^5 &= -i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \end{aligned} \tag{A.2}$$

where σ^m refers to the Pauli matrices. It is easily shown that the 4×4 matrices in (A.1) satisfy the $SO(2,4)$ algebra given in (2.3)

Four dimensional representation of $SO(6)$ generators. The conventions for the four dimensional Weyl representation of $SO(6)$ generators is as follows.

$$\begin{aligned} M^{ij} &= \frac{i}{4}[\gamma^i, \gamma^j], \qquad & M^{i5} &= \frac{i}{2}\gamma^i\gamma^5, \\ M^{i6} &= -\frac{1}{2}\gamma^i, \qquad & M^{56} &= -\frac{1}{2}\gamma^5. \end{aligned} \tag{A.3}$$

Where γ^i and 4×4 $SO(4)$ gamma matrices in the Weyl representation, they are given by

$$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ \bar{\sigma}^i & 0 \end{pmatrix}. \tag{A.4}$$

$\sigma^i = (1, i\vec{\sigma})$ and $\bar{\sigma}^i = (1, -i\vec{\sigma})$. These gamma matrices obey the algebra $\{\gamma^i, \gamma^j\} = 2\delta^{ij}$. Finally γ^5 is given by

$$\gamma^5 = -\gamma^1\gamma^2\gamma^3\gamma^4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.5}$$

JHEP04(2006)014

One can easily verify that the 4×4 matrices of (A.3) satisfy the $SO(6)$ algebra given in (2.25).

Though we use the symbol γ and σ to denote the matrices that occur in both $SO(2,4)$ and $SO(6)$ generators it will be clear from the context and the indices $\mu, \nu, \ldots$ or $i, j, \ldots$ which generator we will be dealing with.

B. Properties of the transform U

The easiest way to obtain the properties of $U = \exp(\frac{\pi}{2} M_{05})$ given in (2.8) is to use the 4×4 representation of $SO(2,4)$ generators. We first verify the relation $U^- D U = iE$. Note that

$$M_{05} = \frac{i}{2}\sigma^2 \otimes 1, \qquad D = -M_{56} = -\frac{i}{2}\sigma^1 \otimes 1, \qquad E = M_{06} = \frac{1}{2}\sigma^3 \otimes 1. \tag{B.1}$$

Therefore U is a $\pi/2$ rotation about the σ^2-axis, given by

$$U = \exp(\frac{\pi}{2} M_{05}) = \frac{1}{\sqrt{2}}(1 + i\sigma^2) \otimes 1, \tag{B.2}$$

using the above expression for U we obtain

$$U^- D U = \exp\left(-\frac{i\pi}{4}\sigma^2\right) \frac{i}{2}\sigma^1 \exp\left(\frac{i\pi}{4}\sigma^2\right) = \frac{i}{2}\sigma^3 = iE. \tag{B.3}$$

The tensor product with the identity 2×2 matrix is understood in the above relation. From the equation (B.3) it easy to see that the property $U^- P_\mu U = L^+$ and $U^- K_\mu U = L^-$ is also satisfied. L^+ is defined as operators which have $+1$ eigen value with the generator E, since E is conjugate to D and P_μ has $+$ eigen value with the operator D, we obtain L^+ is conjugate to P_μ. A similar argument can be applied to the operators L^-. To be more explicit, using $P_\mu = M_{\mu 5} + M_{\mu 6}$ and the 4×4 representation of $SO(2,4)$ generators one can show that

$$U^- P_\mu U = \begin{pmatrix} 0 & \bar{\sigma}_\mu \\ 0 & 0 \end{pmatrix}, \tag{B.4}$$

where $\bar{\sigma}^\mu = (-1, -\vec{\sigma})$. The above relation implies that $\hat{U}^- \hat{P}_\mu \hat{U} = -a^\dagger \bar{\sigma}_\mu b^\dagger$ when one works with the Fock space generators. Similarly it is easily shown that

$$U^- K_\mu K = \begin{pmatrix} 0 & 0 \\ \sigma_\mu & 0 \end{pmatrix}, \tag{B.5}$$

where $\sigma^\mu = (-1, \vec{\sigma})$. This implies that for the Fock space generators we have $\hat{U}^- \hat{K}_\mu \hat{U} = b\sigma_\mu a$.

C. The $PSU(2,2|4)$ algebra

Let us define the following generators for the $PSU(2,2|4)$ algebra

$$A_{\dot{\gamma}\gamma} = a^\dagger_{\dot{\gamma}} b^\dagger_\gamma, \qquad A^{\gamma\dot{\gamma}} = b^\gamma a^{\dot{\gamma}},$$

$$
\begin{aligned}
L^{\dot\delta}_{\dot\gamma} &= a^{\dagger}_{\dot\gamma} a^{\dot\delta} - \frac{1}{2}\delta^{\dot\delta}_{\dot\gamma} N_a, \qquad & R^{\delta}_{\gamma} &= b^{\dagger}_{\gamma} b^{\delta} - \frac{1}{2}\delta^{\delta}_{\gamma} N_b, \\
J^{\tau\dot\tau} &= \alpha^{\tau\dagger}\beta^{\dot\tau\dagger}, \qquad & J_{\dot\tau\tau} &= \beta_{\dot\tau}\alpha_{\tau}, \\
& & L'^{\tau}_{\upsilon} &= \alpha^{\tau\dagger}\alpha_{\upsilon} - \frac{1}{2}\delta^{\tau}_{\upsilon} N_{\alpha}, \\
& & R'^{\dot\tau}_{\dot\upsilon} &= \beta^{\dot\tau\dagger}\beta_{\dot\upsilon} - \frac{1}{2}\delta^{\dot\tau}_{\dot\upsilon} N_{\beta}.
\end{aligned} \tag{C.1}
$$

Then the $PSU(2,2|4)$ algebra is given by the following commutation relations.

$$
\begin{aligned}
[A^{\delta\dot\delta}, A_{\dot\gamma\gamma}] &= \delta^{\delta}_{\gamma} L^{\dot\delta}_{\dot\gamma} + \delta^{\dot\delta}_{\dot\gamma} R^{\delta}_{\gamma} + \delta^{\delta}_{\gamma}\delta^{\dot\delta}_{\dot\gamma} E, \\
[J^{\tau\dot\tau}, J_{\upsilon\dot\upsilon}] &= \delta^{\dot\tau}_{\dot\upsilon} L'^{\tau}_{\upsilon} + \delta^{\tau}_{\upsilon} R'^{\dot\tau}_{\dot\upsilon} - \delta^{\tau}_{\upsilon}\delta^{\dot\tau}_{\dot\upsilon} J, \\
\{Q^{-\ \dot\tau}_{\dot\gamma}, S^{+\ \dot\delta}_{\dot\upsilon}\} &= \delta^{\dot\tau}_{\dot\upsilon} L^{\dot\delta}_{\dot\gamma} - \delta^{\dot\delta}_{\dot\gamma} R'^{\dot\tau}_{\dot\upsilon} + \frac{1}{2}\delta^{\dot\tau}_{\dot\upsilon}\delta^{\dot\delta}_{\dot\gamma}(E+J+Z), \\
\{Q^{-\ \tau}_{\gamma}, S^{+\ \delta}_{\upsilon}\} &= \delta^{\tau}_{\upsilon} R^{\delta}_{\gamma} - \delta^{\delta}_{\gamma} L'^{\tau}_{\upsilon} + \frac{1}{2}\delta^{\tau}_{\upsilon}\delta^{\delta}_{\gamma}(E+J+Z), \\
\{Q^{+}_{\dot\gamma\tau}, S^{-\ \dot\delta\upsilon}\} &= \delta^{\upsilon}_{\tau} L^{\dot\delta}_{\dot\gamma} + \delta^{\dot\delta}_{\dot\gamma} L'^{\upsilon}_{\tau} + \frac{1}{2}\delta^{\dot\delta}_{\dot\gamma}\delta^{\upsilon}_{\tau}(E-J+Z), \\
\{Q^{+}_{\gamma\dot\tau}, S^{-\ \delta\dot\upsilon}\} &= \delta^{\dot\upsilon}_{\dot\tau} R^{\delta}_{\gamma} + \delta^{\delta}_{\gamma} R'^{\dot\upsilon}_{\dot\tau} + \frac{1}{2}\delta^{\delta}_{\gamma}\delta^{\dot\upsilon}_{\dot\tau}(E-J+Z).
\end{aligned} \tag{C.2}
$$

Here the Q's are defined in table 2. Note that the generator $B = N_\alpha - N_\beta$ does not appear on the left hand side of the (anti)-commutator relations, therefore it acts as an external automorphism of the algebra.

D. Bit string overlap and three-point functions

Length conserving process. We will first consider examples of various length conserving processes to test the vertex in (3.41). Consider the following Yang-Mill states

$$
\begin{aligned}
O^{(1)} &= \frac{1}{\sqrt{N^{l_1+1}}}\mathrm{Tr}(\phi^i z^{l_1}), \qquad O^{(2)} = \frac{1}{\sqrt{N^{l_2+1}}}\mathrm{Tr}(\phi^j z^{l_2}), \\
O^{(3)} &= \frac{1}{\sqrt{N^{l_1+l_2+2}}}\mathrm{Tr}(\phi^i \bar z^J \phi^j \bar z^{l_1+l_2-l}), \qquad i \neq j, l_1 < l_2.
\end{aligned} \tag{D.1}
$$

The normalizations are chosen such that in the large N limit the leading term in their two point function is canonically normalized. The two point functions of the fields involved are given by

$$
\phi^i_{ab}(x_1)\phi^j_{a'b'}(x_2) = \frac{\delta_{ij}\delta_{aa'}\delta_{bb'}}{|x_1-x_2|^2}, \qquad z_{ab}(x_1) z_{a'b'}(x_2) = \frac{\delta_{aa'}\delta_{bb'}}{|x_1-x_2|^2}, \tag{D.2}
$$

where a, b label the $U(N)$ indices. The three point functions of these operators in the planar limit is given by

$$
\begin{aligned}
\langle O^{(1)}(x_1) O^{(2)}(x_2) O^{(3)}(x_3)\rangle &= \frac{l+1}{N|x_{23}|^{2(l_2+1)}|x_{31}|^{2(l_1+1)}}, \quad l < l_1, \\
&= \frac{l_1+1}{N|x_{23}|^{2(l_2+1)}|x_{31}|^{2(l_1+1)}}, \quad l_1 \le l \le l_2, \\
&= \frac{l_1+l_2-l+1}{N|x_{23}|^{2(l_2+1)}|x_{31}|^{2(l_1+1)}}, \quad l > l_2,
\end{aligned} \tag{D.3}
$$

where $x_{13} = x_1 - x_2$, $x_{23} = x_2 - x_3$. The multiplicity occurring in the structure constants are due to the cyclicity of the trace. To evaluate this correlation function using the three string vertex, we first set up the dictionary of states.

$$\begin{aligned}
\langle O^{(1)}(x_1)| &= {}^{(1)}\langle l_1+1| \sum_{s=0}^{l_1} \frac{1}{\sqrt{2}} (\beta\bar{\sigma}^i\alpha)(s), \\
\langle O^{(2)}(x_2)| &= {}^{(2)}\langle l_2+1| \sum_{u=l_1+1}^{l_1+l_2+1} \frac{1}{\sqrt{2}} (\beta\bar{\sigma}^i\alpha)(u), \\
\langle O^{(3)}(x_3)| &= {}^{(3)}\langle l_1+l_2+2| \sum_{t=0}^{l_1+l_2+1} \frac{1}{2} (\beta\bar{\sigma}^i\alpha(t)(\beta\bar{\sigma}^j\alpha)(t+l+1).
\end{aligned} \tag{D.4}$$

Here the state ${}^{(1)}\langle l_1+1|$ refers to the l_1+1 bit vacuum state at position x_1, explicitly it is given by

$${}^{(1)}\langle l_1+1| = \langle 0| \exp(\sum_{s=0}^{l_1+1} ix_1 P^\dagger(s)). \tag{D.5}$$

Similar definitions apply for the states ${}^2\langle l_2+1|$ and ${}^3\langle l_1+l_2+2|$. Evaluating the three string vertex on these states we obtain

$$\begin{aligned}
\langle O^{(1)}| \otimes \langle O^{(2)}| \otimes \langle O^{(3)}| \;\; |V_3\rangle &= \sum_{t=0}^{l_1+l_2+1} \sum_{u=l_1+1}^{l_1+l_2+1} \sum_{s=0}^{l_1} \delta(s,t)\delta(u,t+l+1) \\
&\quad \times \frac{1}{N|x_{23}|^{2(l_2+1)}|x_{31}|^{2(l_1+1)}}, \\
&= \frac{l+1}{N|x_{23}|^{2(l_2+1)}|x_{31}|^{2(l_1+1)}}, \quad l < l_1, \\
&= \frac{l_1+1}{N|x_{23}|^{2(l_2+1)}|x_{31}|^{2(l_1+1)}}, \quad l_1 \le l \le l_2, \\
&= \frac{l_1+l_2-l+1}{N|x_{23}|^{2(l_2+1)}|x_{31}|^{2(l_1+1)}}, \quad l > l_2.
\end{aligned} \tag{D.6}$$

Thus the overlap rules of the three string vertex agrees with the gauge theory correlator given in (D.3).

Length non-conserving processes. As an example for a length non-conserving process, consider the correlation function of

$$\begin{aligned}
O^{(1)}(x_1) &= \frac{1}{\sqrt{N^{l_1+2}}} \mathrm{Tr}(\phi^j z^l \phi^i z^{l_1-l}), \\
O^{(2)}(x_2) &= \frac{1}{\sqrt{N^{l_2+2}}} \mathrm{Tr}(\phi^i z^m \phi^k z^{l_1-m}), \\
O^{(3)}(x_3) &= \frac{1}{\sqrt{N^{l_3+2}}} \mathrm{Tr}(\phi^j \bar{z}^n \phi^k \bar{z}^{l_1-n}), \quad i \ne j \ne k, \;\; l_1+l_2 = l_3.
\end{aligned} \tag{D.7}$$

The correlation function of the three operators in the large N limit is given by

$$\langle O^{(1)}(x_1) O^{(2)}(x_2) O^{(3)}(x_3)\rangle = \frac{\delta(l+m,n)}{N} \frac{1}{|x_{12}|^2 |x_{23}|^{2(l_2+1)} |x_{13}|^{2(l_1+1)}} \tag{D.8}$$

For this process, the length violation is 2 units. To evaluate the correlation function in string variables we set up a dictionary similar to the previous case.

$$\langle O^{(1)}| = \sum_{s=0}^{l_1+1} {}^{(1)}\langle l_1+2|(\beta\bar{\sigma}^j\alpha)(s)(\beta\bar{\sigma}^i\alpha)(s+l+1), \tag{D.9}$$
$$\langle O^{(2)}| = \sum_{t=l_1+1}^{l_3+2} {}^{(2)}\langle l_2+2|(\beta\bar{\sigma}^i\alpha)(t)(\beta\bar{\sigma}^k\alpha)(t+m+1),$$
$$\langle O^{(3)}| = \sum_{u=0}^{l_3+1} {}^{(3)}\langle l_3+2|(\beta\bar{\sigma}^j\alpha)(u)(\beta\bar{\sigma}^k\alpha)(u+n+1).$$

Evaluating the length non-conserving vertex in (3.43) with $l=1$ on these states we obtain.

$$\begin{aligned}
&\langle O^{(1)}| \otimes \langle O^{(2)}| \otimes \langle O^{(3)}| \ \ |V_3\rangle = \\
&\sum_{u=0}^{l_3+1}\sum_{s=0}^{l_1+1}\left(\sum_{t=l_1+1}^{l_3+2} \delta(s,u)\delta(s+l+1,l_1+1)\delta(t,l_1+1)\right. \\
&\left.\delta(t+m+1,u+s+n+2)\right) \times \frac{1}{N|x_{12}|^2|x_{23}|^{2(l_2+1)}|x_{13}|^{2(l_1+1)}}, \\
= &\frac{\delta(m+l,n)}{N|x_{12}|^2|x_{23}|^{2(l_2+1)}|x_{13}|^{2(l_1+1)}}.
\end{aligned} \tag{D.10}$$

Here the two delta functions with l_1+1 in the argument are due to the fact that the 1 and 2 strings overlap only if the bits at the l_1+1 sites are equal. The argument in the last delta function $u+s+n+2$ is due to the fact the overlap of the 2nd and 3rd string are shifted by $l=1$ in (3.43). Thus there is agreement with the field theory calculation (D.8). The above calculation can easily be extended to cases with $i=j$ etc.

E. Anomalous dimensions at three loops

As we mentioned in section 5, the model given in (5.2) is not sufficient to obtain the anomalous dimensions at three loops. In this appendix we show that it is possible to reproduce the anomalous dimensions to three loops for the $SU(2)$ subsector using a gauged linear oscillator model, but with bilinears of oscillators at positions $(s,s+2)$, $(s,x+3)$ in addition to oscillators at $(s,s+1)$. Let us define the anomalous dimension Hamiltonian to be given by

$$\exp(c\sum_{n=1}^{3}\lambda^n H^{(n)}) = \int \mathcal{D}U \exp(\sum_{n=12,3,s}^{3} g^n f_n(A_s^{(n)}U_{s,s+n} + \text{h.c.})), \tag{E.1}$$

where $g=\sqrt{\lambda}$ and f_n are functions of the 't Hooft coupling λ, with the expansions given by

$$\begin{aligned}
f_1 &= f_{10} + \lambda f_{11} + \lambda^2 f_{12}, \\
f_2 &= f_{20} + \lambda f_{21}, \\
f_3 &= f_{30}.
\end{aligned} \tag{E.2}$$

JHEP04(2006)014

In these expansions we have assumed that the couplings f_n are themselves expansions of in the 't Hooft coupling λ. $U_{s,s+n}$ is a link variable which transforms as $U_{s,s+n} \to \exp(i\theta_s)U_{s,s+n}\exp(-i\theta_{s+n})$ under the local $U(1)_Z$ transformation. The anomalous dimensions to three loops is given by [44]

$$\begin{aligned} H^{(1)} = a_1, \qquad H^{(2)} = a_2 - 4a_1, \\ H^{(3)} = 2a_3 - 12a_2 + 30a_1 + t_2 - t_3, \end{aligned} \tag{E.3}$$

where the a_n and t_n are defined by

$$\begin{aligned} a_n &= (1 - \sigma_s \cdot \sigma_{s+n}), \\ t_1 &= (1 - \sigma_s \cdot \sigma_{s+1})(1 - \sigma_{s+2} \cdot \sigma_{s+3}), \\ t_2 &= (1 - \sigma_s \cdot \sigma_{s+2})(1 - \sigma_{s+1} \cdot \sigma_{s+3}), \\ t_3 &= (1 - \sigma_s \cdot \sigma_{s+3})(1 - \sigma_{s+1} \cdot \sigma_{s+2}). \end{aligned} \tag{E.4}$$

In the above equations summation over the sites s is implied. Now our strategy will be to perform the integrations over the link variables in (E.1) and determine the f_{mn} and c so that the anomalous dimensions to three loops, known from gauge theory calculations, is reproduced. At this point one might think that there is a sufficient number of unknowns in f_{mn} and c to allow to obtain $H^{(n)}$. However we will see that the f_{mn} are overdetermined but they admit a unique consistent solution. We now expand the exponential in (E.1) to the sixth order. Here we write the expressions obtained after the integrations over the link variables in (E.1)

$$\begin{aligned} \tilde{H} &= \frac{1}{2}\left(g^2 f_1^2 a_1 + g^4 f_2^2 a_2 + g^6 f_3^2 a_3\right) \\ &+ \frac{3}{4!} f_1^4 g^4 \left(-4a_1 + a_2\right) + \left\{ \frac{g^4}{2}\left(\frac{f_1^2 a_1}{2}\right)^2 \right\} \\ &+ \frac{1}{4!} f_3 f_1^3 g^6 \left[12a_1 - 12a_2 + 4a_3 + 6(-t_1 + t_2 - t_3)\right] \\ &+ \frac{6}{4!} f_1^2 f_2^2 g^6 \left(-t_1 - t_2 + t_3\right) + \\ &+ \frac{1}{6!} g^3 f_1^6 \left(40(a_3 - 6a_2 + 15a_1) + 15(t_1 + t_2 - t_3)\right) \\ &+ \left\{ g^6 \frac{f_1^2 a_1}{2} \frac{3 f_1^4}{4!}(-4a_1 + a_2) + g^6 \left(\frac{f_1^2 a_1}{2}\right)\left(\frac{f_2^2 a_2}{2}\right) \right\}. \end{aligned} \tag{E.5}$$

To obtain the above equation we have used various identities from Appendix F and performed similar manipulations as for the two loop case discussed in section 5. Note that in (E.5), the non-local terms are written in curly bracket. They correspond to the exponentiation of the lower order terms. For instance the first term in the curly brackets correspond to the exponentiation of the one-loop term $g^2 f_1^2 a_1/2$. Similar arguments hold for the other non-local terms. Therefore on exponentiating, as in (E.1), we will obtain a local Hamiltonian.

We will now deal only with the local terms and impose the condition that the local terms organize themselves to be the anomalous dimensions Hamiltonian to 3-loops given in (E.3). Note that the coefficient of t_1 for the anomalous dimension at 3-loop is zero, this

JHEP04(2006)014

implies the following equation for the couplings

$$1 - 12\frac{f_{30}}{f_{10}^3} - 12\left(\frac{f_{20}}{f_{10}^2}\right)^2 = 0. \tag{E.6}$$

We also have the condition that the coefficient of a_3 is twice the coefficient of t_2, this leads to the following equation

$$12\left(\frac{f_{30}}{f_{10}^3}\right)^2 + 4\frac{f_{30}}{f_{10}^3} + \frac{4}{3} = 12\frac{f_{30}}{f_{10}^3} - 12\left(\frac{f_{20}}{f_{10}^2}\right)^2 + 1. \tag{E.7}$$

Combining (E.6) and (E.7) we obtain a quadratic equation for the ratio f_{30}/f_{10}^3.

$$\begin{aligned} 12\left(\frac{f_{30}}{f_{10}^3}\right)^2 - 20\frac{f_{30}}{f_{10}^3} + \frac{4}{3} &= 0, \\ \frac{f_{30}}{f_{10}^3} &= \frac{5 \pm \sqrt{21}}{6}. \end{aligned} \tag{E.8}$$

Among the two roots we choose the latter one as that ensures the ratio f_{20}/f_{10}^2 to be real. Substituting this in (E.6) we obtain

$$\left(\frac{f_{20}}{f_{10}^2}\right)^2 = \frac{2\sqrt{21} - 9}{12}. \tag{E.9}$$

There are two further conditions to be met so that the result of the integrations over the link variables result in the anomalous dimension to 3-loops: (i) The ratio of the coefficient of the g^2 term to that of a_2 in the g^4 term must be one; (ii) the ratio of the coefficient of the t_2 to that of the g^2 term also should be one. These two conditions ensure that three loop anomalous dimension is obtained upto an overall scaling constant $c = f_{10}^2/2$ in the exponential of (E.1) These conditions give two independent equations for the coupling f_{10} which admit a common solution. The first condition gives

$$\frac{1}{f_{10}^2} = \left(\frac{f_{20}}{f_{10}}\right)^2 + \frac{1}{4} = \frac{\sqrt{21} - 3}{6}, \tag{E.10}$$

while the second condition gives

$$\begin{aligned} \frac{1}{4!}\left(6\frac{f_{30}}{f_{10}^3} - 6\left(\frac{f_{10}}{f_{20}}\right)^2 + \frac{1}{2}\right) &= \frac{1}{2f_{10}^2}\left(\left(\frac{f_{20}}{f_{10}^2}\right)^2 + \frac{1}{4}\right), \\ &= \frac{1}{2}\frac{1}{f_{10}^4}. \end{aligned} \tag{E.11}$$

Substituting the equation (E.6) in the left hand side of the first line we obtain the consistency condition

$$\frac{f_{30}}{f_{10}^3} = \left(\frac{1}{f_{10}^2}\right)^2. \tag{E.12}$$

This condition is clearly satisfied by the solution in latter solution in (E.8) and (E.11). We now can easily fix the remaining constants f_{11}, f_{12}, f_{21} by requiring BMN scaling. Though we have obtained the anomalous dimensions to three loop, the model in (E.1) is not entirely satisfactory. This is because we do not have a principle to determine the constants f_n, further more we have not taken the modified commutation relations of section 4. into account. Taking this into account might change the values of the f_n's found in this appendix.

F. Properties of bilinears in the $SU(2)$ sector

We define a oscillator bilinear of length l to be given by

$$A_s^{(l)} = \bar{\varphi}_s \varphi_{s+l} = \alpha_s^\dagger \alpha_{s+l} + \beta_s \beta_{s+1}^\dagger. \tag{F.1}$$

Then the various commutation relations which are required to obtain the anomalous dimension Hamiltonian upto three loops are given by

$$\begin{aligned}
[A_s^{(1)}, A_t^{(1)\dagger}] &= \delta_{st}(N_s^\alpha - N_s^\beta - (N_{s+1}^\alpha - N_{s+1}^\beta)) = \delta_{st} G_s, \\
[G_s, A_t^{(1)\dagger}] &= (\delta_{s,t+1} + \delta_{t,s+1} - 2\delta_{st}) A_t^{(1)\dagger}, \\
[A_s^{(1)}, A_t^{(1)}] &= \delta_{s+1,t} A_s^{(2)} - \delta_{s,t+1} A_t^{(2)}, \\
[A_s^{(1)\dagger}, A_t^{(1)\dagger}] &= \delta_{t+1,s} A_t^{(2)\dagger} - \delta_{t,s+1} A_s^{(2)\dagger}, \\
[A_s^{(2)}, A_t^\dagger] &= \delta_{s+1,t} A_s^{(1)} - \delta_{st} A_{s+1}^{(1)}, \\
[A_s, A_t^{(2)}] &= \delta_{s+1,t} A_s^{(3)} - \delta_{s,t+2} A_s^{(3)}, \\
[A_s^{(1)}, A_t^{(2)\dagger}] &= \delta_{s,t+1} A_t^{(1)\dagger} - \delta_{st} A_{t+1}^{(1)\dagger}, \\
[A_s^{(3)}, A_t^{(1)\dagger}] &= \delta_{s+2,t} A_x^{(2)} - \delta_{st} A_{s+1}^{(2)}.
\end{aligned} \tag{F.2}$$

One also can obtain other commutation relations by taking the Hermitian conjugates of the above.

The identities below relate the bilinears to Pauli matrices, these relations are always valid on physical states which satisfy the $U(1)_Z$ constraint. Let us define then we have

$$\begin{aligned}
A_s^{(l} A_s^{(l)\dagger} &= \frac{1}{2}(1 - \sigma_s \cdot \sigma_{s+l}), \\
A_s^{(l)\dagger} A_s^{(l)} &= \frac{1}{2}(1 - \sigma_s \cdot \sigma_{s+1}), \\
A_s^{(2)} A_s^{(1)\dagger} A_{s+1}^{(1)\dagger} &= -\frac{1}{4}(1 - \sigma_s \cdot \sigma_{s+2})(1 - \sigma_{s+1} \cdot \sigma_{s+2}), \\
A_{s+1}^{(2)} A_s^{(1)} A_s^{(3)\dagger} &= -\frac{1}{4}(1 - \sigma_{s+1} \cdot \sigma_{s+3})(1 - \sigma_s \cdot \sigma_{s+3}), \\
A_s^{(2)} A_{s+2}^{(1)} A_s^{(3)\dagger} &= \frac{1}{4}(1 - \sigma_{s+1} \cdot \sigma_{s+2})(1 - \sigma_s \cdot \sigma_{s+3}), \\
A_s^{(2)\dagger} A_s^{(1)} A_{s+1}^{(1)} &= \frac{1}{4}(1 - \sigma_s \cdot \sigma_{s+2})(1 - \sigma_{s+1} \cdot \sigma_{s+2}), \\
A_{s+1}^{(2)\dagger} A_s^{(1)\dagger} A_s^{(3)} &= \frac{1}{4}(1 - \sigma_{s+1} \cdot \sigma_{s+3})(1 - \sigma_s \cdot \sigma_{s+3}), \\
A_s^{(2)\dagger} A_{s+2}^{(1)\dagger} A_s^{(3)} &= -\frac{1}{4}(1 - \sigma_s \cdot \sigma_{s+2})(1 - \sigma_s \cdot \sigma_{s+3}).
\end{aligned} \tag{F.3}$$

In all the above equations and the remaining equations summation over the site label s is implied. The identities below relate 4 bilinears to quartic terms in Pauli matrices which are are different sites. We first define the various quartic terms in Pauli matrices

$$t_1 = (1 - \sigma_s \cdot \sigma_{s+1})(1 - \sigma_{s+2} \cdot \sigma_{s+3}), \tag{F.4}$$
$$t_2 = (1 - \sigma_s \cdot \sigma_{s+2})(1 - \sigma_{s+1} \cdot \sigma_{s+3}),$$
$$t_3 = (1 - \sigma_s \cdot \sigma_{s+3})(1 - \sigma_{s+1} \cdot \sigma_{s+2}).$$

We have the following identities

$$A^{(2)}_{s+1} A^{(2)\dagger}_s A^{(1)}_s A^{(1)\dagger}_{s+1} + \text{h.c.} = \frac{1}{4}(t_3 - t_2 - t_1),$$
$$A^{(3)}_s A^{(1)\dagger}_{s+1} A^{(1)\dagger}_s A^{(1)\dagger}_{s+2} + \text{h.c} = \frac{1}{4}(-t_3 + t_2 - t_1). \tag{F.5}$$

We also have the following identity

$$A^{(2)}_s A^{(1)}_{s+1} A^{(1)\dagger}_{s+1} A^{(2)\dagger}_s + A^{(2)}_s A_s A^\dagger_s A^{(2)\dagger}_s = 0. \tag{F.6}$$

Note that in all manipulation whenever the Gauss law G_s occurs at either the extreme left or right of a string of bilinears one can set that term to zero as it vanishes on any physical state. Also whenever higher $A_s A_s$ or higher powers and $A^\dagger_s A^\dagger_s$ or higher powers occur at either the extreme left or right that term can be set to zero as one can easily verify it vanishes on any physical state.

References

[1] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200].

[2] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150].

[3] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B 428** (1998) 105 [hep-th/9802109].

[4] O. Aharony, S.S. Gubser, J.M. Maldacena, H. Ooguri and Y. Oz, *Large-N field theories, string theory and gravity*, *Phys. Rept.* **323** (2000) 183 [hep-th/9905111].

[5] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *A semi-classical limit of the gauge/string correspondence*, *Nucl. Phys.* **B 636** (2002) 99 [hep-th/0204051].

[6] S.-M. Lee, S. Minwalla, M. Rangamani and N. Seiberg, *Three-point functions of chiral operators in $D = 4$, $N = 4$ SYM at large-N*, *Adv. Theor. Math. Phys.* **2** (1998) 697 [hep-th/9806074].

[7] D.Z. Freedman, S.D. Mathur, A. Matusis and L. Rastelli, *Correlation functions in the CFT(d)/AdS(d + 1) correspondence*, *Nucl. Phys.* **B 546** (1999) 96 [hep-th/9804058].

[8] G. Arutyunov and S. Frolov, *Some cubic couplings in type-IIB supergravity on $AdS_5 \times S^5$ and three-point functions in SYM(4) at large-N*, *Phys. Rev.* **D 61** (2000) 064009 [hep-th/9907085].

[9] D. Berenstein, J.M. Maldacena and H. Nastase, *Strings in flat space and pp waves from $N = 4$ super Yang-Mills*, *JHEP* **04** (2002) 013 [hep-th/0202021].

JHEP04(2006)014

[10] G. Arutyunov, S. Frolov, J. Russo and A.A. Tseytlin, *Spinning strings in $AdS_5 \times S^5$ and integrable systems*, *Nucl. Phys.* **B 671** (2003) 3 [hep-th/0307191].

[11] G. Arutyunov, S. Frolov and M. Staudacher, *Bethe ansatz for quantum strings*, *JHEP* **10** (2004) 016 [hep-th/0406256].

[12] J. Plefka, *Spinning strings and integrable spin chains in the AdS/CFT correspondence*, hep-th/0507136.

[13] M. Bianchi, J.F. Morales and H. Samtleben, *On stringy $AdS_5 \times S^5$ and higher spin holography*, *JHEP* **07** (2003) 062 [hep-th/0305052].

[14] N. Beisert, M. Bianchi, J.F. Morales and H. Samtleben, *On the spectrum of AdS/CFT beyond supergravity*, *JHEP* **02** (2004) 001 [hep-th/0310292].

[15] M. Bianchi, *Higher spins and stringy $AdS_5 \times S^5$*, *Fortschr. Phys.* **53** (2005) 665 [hep-th/0409304].

[16] R. Gopakumar, *From free fields to AdS*, *Phys. Rev.* **D 70** (2004) 025009 [hep-th/0308184].

[17] R. Gopakumar, *From free fields to AdS, II*, *Phys. Rev.* **D 70** (2004) 025010 [hep-th/0402063].

[18] R. Gopakumar, *From free fields to AdS, III*, *Phys. Rev.* **D 72** (2005) 066008 [hep-th/0504229].

[19] A. Dhar, G. Mandal and S.R. Wadia, *String bits in small radius AdS and weakly coupled $N = 4$ super Yang-Mills theory, I*, hep-th/0304062.

[20] R.R. Metsaev, *Light cone gauge formulation of IIB supergravity in $AdS_5 \times S^5$ background and AdS/CFT correspondence*, *Phys. Lett.* **B 468** (1999) 65 [hep-th/9908114].

[21] R.R. Metsaev, C.B. Thorn and A.A. Tseytlin, *Light-cone superstring in AdS space-time*, *Nucl. Phys.* **B 596** (2001) 151 [hep-th/0009171].

[22] A.A. Tseytlin, *On limits of superstring in $AdS_5 \times S^5$*, *Theor. Math. Phys.* **133** (2002) 1376 [hep-th/0201112].

[23] N.R. Constable et al., *PP-wave string interactions from perturbative Yang-Mills theory*, *JHEP* **07** (2002) 017 [hep-th/0205089].

[24] C. Kristjansen, J. Plefka, G.W. Semenoff and M. Staudacher, *A new double-scaling limit of $N = 4$ super Yang-Mills theory and pp-wave strings*, *Nucl. Phys.* **B 643** (2002) 3 [hep-th/0205033].

[25] H.L. Verlinde, *Bits, matrices and 1/N*, *JHEP* **12** (2003) 052 [hep-th/0206059].

[26] D. Vaman and H.L. Verlinde, *Bit strings from $N = 4$ gauge theory*, *JHEP* **11** (2003) 041 [hep-th/0209215].

[27] J.-G. Zhou, *pp-wave string interactions from string bit model*, *Phys. Rev.* **D 67** (2003) 026010 [hep-th/0208232].

[28] K. Okuyama and L.-S. Tseng, *Three-point functions in $N = 4$ SYM theory at one-loop*, *JHEP* **08** (2004) 055 [hep-th/0404190].

[29] R. Roiban and A. Volovich, *Yang-Mills correlation functions from integrable spin chains*, *JHEP* **09** (2004) 032 [hep-th/0407140].

[30] L.F. Alday, J.R. David, E. Gava and K.S. Narain, *Structure constants of planar $N = 4$ Yang-Mills at one loop*, *JHEP* **09** (2005) 070 [hep-th/0502186].

[31] M. Günaydin and N. Marcus, *The spectrum of the S^5 compactification of the chiral $N = 2$, $D = 10$ supergravity and the unitary supermultiplets of U(2, 2/4)*, *Class. and Quant. Grav.* **2** (1985) L11.

[32] M. Günaydin, D. Minic and M. Zagermann, *4d doubleton conformal theories, CPT and IIB string on $AdS_5 \times S^5$*, *Nucl. Phys.* **B 534** (1998) 96 [hep-th/9806042].

[33] B. Sundborg, *The Hagedorn transition, deconfinement and $N = 4$ SYM theory*, *Nucl. Phys.* **B 573** (2000) 349 [hep-th/9908001].

[34] A.M. Polyakov, *Gauge fields and space-time*, *Int. J. Mod. Phys.* **A17S1** (2002) 119–136 [hep-th/0110196].

[35] O. Aharony, J. Marsano, S. Minwalla, K. Papadodimas and M. Van Raamsdonk, *The hagedorn/deconfinement phase transition in weakly coupled large-N gauge theories*, *Adv. Theor. Math. Phys.* **8** (2004) 603 [hep-th/0310285].

[36] J. Wess and J. Bagger, *Supersymmetry and supergravity*, Princeton, USA, Univ. Pr., Princeton, 1992, p. 259.

[37] S. Ferrara, S. Gatto, and A. Grillo, *Conformal algebra in space-time and operator product expansion*, Springer-Verlag, Berlin, 1973,p. 63.

[38] T. Kugo and B. Zwiebach, *Target space duality as a symmetry of string field theory*, *Prog. Theor. Phys.* **87** (1992) 801 [hep-th/9201040].

[39] N. Beisert, *The complete one-loop dilatation operator of $N = 4$ super Yang-Mills theory*, *Nucl. Phys.* **B 676** (2004) 3 [hep-th/0307015].

[40] P.A. Dirac, *Lectures on quantum mechanics*, Belfer Graduate School of Sciences, Yeshiva University, New York, 1964.

[41] I. Affleck, Z. Zou, T. Hsu, and P. W. Anderson, *$SU(2)$ gauge symmetry of the large-U Hubbard model*, *Phys. Rev.* **38** (1988) 645.

[42] E. Dagotto, E.H. Fradkin, and A. Moreo, *$SU(2)$ gauge invariance and order parameters in strongly coupled electronic systems*, *Phys. Rev.* **B 38** (1988) 2926.

[43] N. Beisert, C. Kristjansen and M. Staudacher, *The dilatation operator of $N = 4$ super Yang-Mills theory*, *Nucl. Phys.* **B 664** (2003) 131 [hep-th/0303060].

[44] N. Beisert, *The su(2—3) dynamic spin chain*, *Nucl. Phys.* **B 682** (2004) 487 [hep-th/0310252].

[45] B. Eden, C. Jarczak and E. Sokatchev, *A three-loop test of the dilatation operator in $N = 4$ SYM*, *Nucl. Phys.* **B 712** (2005) 157 [hep-th/0409009].

[46] J.A. Minahan and K. Zarembo, *The bethe-ansatz for $N = 4$ super Yang-Mills*, *JHEP* **03** (2003) 013 [hep-th/0212208].

[47] C.G. Callan Jr. et al., *Quantizing string theory in $AdS_5 \times S^5$: beyond the pp-wave*, *Nucl. Phys.* **B 673** (2003) 3 [hep-th/0307032].

[48] C.G. Callan Jr., T. McLoughlin and I.J. Swanson, *Holography beyond the Penrose limit*, *Nucl. Phys.* **B 694** (2004) 115 [hep-th/0404007].

[49] C.G. Callan Jr., T. McLoughlin and I.J. Swanson, *Higher impurity AdS/CFT correspondence in the near-BMN limit*, *Nucl. Phys.* **B 700** (2004) 271 [hep-th/0405153].

[50] T. McLoughlin and I.J. Swanson, *N-impurity superstring spectra near the pp-wave limit*, *Nucl. Phys.* **B 702** (2004) 86 [hep-th/0407240].

[51] L.F. Alday, G. Arutyunov and S. Frolov, *New integrable system of 2DIM fermions from strings on* $AdS_5 \times S^5$, *JHEP* **01** (2006) 078 [hep-th/0508140].